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Discrete uncertainty principles and Virial identities

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ABSTRACT

In this paper we review the Heisenberg uncertainty principle in a discrete setting and, as in the classical uncertainty principle, we give it a dynamical sense related to the discrete Schrödinger equation. We study the convergence of the relation to the classical uncertainty principle, and, as a counterpart, we also obtain another discrete uncertainty relation that does not have an analogous form in the continuous case. Moreover, in the case of the Discrete Fourier Transform, we give an inequality that allows us to relate the minimizer to the Gaussian.

1. Introduction

The well-known Heisenberg uncertainty principle [4] states that

$$\frac{2}{d}\left(\int_{\mathbb{R}^d} |xf(x)|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^d} |\nabla f(x)|^2\,dx\right)^{1/2} \geq \int_{\mathbb{R}^d} |f(x)|^2\,dx. \tag{1}$$

Moreover, the minimizing function (that for which (1) is an equality) satisfies, for $\alpha > 0$, $\nabla f(x)+\alpha x f(x) = 0 \implies f(x) = Ce^{-\alpha|x|^2/2}$ (Gaussian).

Now, if we consider $u(x,t)$ a solution to the Schrödinger free equation, there is a dynamic interpretation of the uncertainty principle, which was exploited in [5,6].

Theorem 1.1 *(Dynamic uncertainty principle). Assume $u(x,t)$ is a solution to*

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$$\begin{cases} \partial_t u(x,t) = i\Delta u(x,t), & x \in \mathbb{R}^d, t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases}$$

where $u_0 \in \dot{H}^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, |x|^2\,dx)$, $\|u_0\|_2^2 = 2/d$. *For a real function* $\phi(x)$ *we define*

$$h(t) = \int_{\mathbb{R}^d} \phi(x)|u(x,t)|^2\,dx, \quad a = \int_{\mathbb{R}^d} |x|^2|u_0(x)|^2\,dx < +\infty, \quad b = \int_{\mathbb{R}^d} |\nabla u_0(x)|^2\,dx < +\infty.$$

Then,

$$\ddot{h}(t) = 4\int_{\mathbb{R}^d} \nabla u D^2\phi \overline{\nabla u} - \int_{\mathbb{R}^d} \Delta^2\phi|u|^2. \quad (\textit{Virial identity}) \tag{2}$$

Moreover, if $\phi(x) = |x|^2$,

$$h(t) = a + 4bt^2 \geq a + \frac{4t^2}{a}, \tag{3}$$

and, if these two convex parabolas intersect each other, they are the same parabola and the initial datum is $u_0(x) = Ce^{-\alpha|x|^2/2}$, *being then*

$$u(x,t) = \left(\frac{1}{2\alpha it + 1}\right)^{d/2} \exp\left(\frac{i\alpha^2 t|x|^2}{4\alpha^2t^2+1} - \frac{\alpha|x|^2}{8\alpha^2t^2+2}\right).$$

Observe that the normalization condition in the initial datum gives, thanks to the uncertainty principle (1) that $ab \geq 1$.

In this paper we want to develop this theory in a discrete setting discretizing the momentum and position operators. Since we can relate a sequence to a periodic function via Fourier series, there is a duality between discrete uncertainty principles and periodic uncertainty principles. The relation we study here appears in the literature (see [2,8,3]) in this periodic form. Moreover, in [3] the authors suggested another uncertainty relation. Their aim was to study the angular momentum – angle variables on the sphere, so they related the orbital angular momentum to the azimuthal angle about the z axis. Then, the orbital momentum is written as a differential operator and, for a meaningful uncertainty principle, periodicity is required for the position operator. Hence, the authors suggested the operators $\cos(x)$ and $\sin(x)$ to represent position. Considering this duality via Fourier series, the second case is connected with the discrete version of Heisenberg uncertainty principle that we will study here. In the first case, we will get another relation that does not have a continuous version.

Another version of the Heisenberg uncertainty principle appears in [12,13,8], but in this case the equality is not attained. However, it is possible to construct a sequence of polynomials p_k of degree k such that the inequality approaches the equality as k tends to infinity. Nevertheless, we will not study this relation here.

As it happens for the Heisenberg uncertainty principle in the continuous case, we will derive Virial identities equivalent to (2) for both relations. Thus we will give them a dynamical interpretation (equivalent to (3)). On the one hand, the dynamics will be given by the discrete Schrödinger equation, as it is expected. On the other hand, it will appear an equation that turns out to be an L^2-invariant factorization of the one dimensional wave equation.

Since we see an analogy between the continuous and discrete dynamic uncertainty principles, it seems reasonable to have similarities between the solution to the continuous Schrödinger equation with initial datum the Gaussian and the solution to the discrete equation, now with initial datum the minimizer of the discrete relation. In the continuous case, it is known that this solution satisfies another equation in the form