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Window-dependent bases for efficient representations of the Stockwell transform

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ABSTRACT

Since its appearing in 1996, the Stockwell transform (S-transform) has been applied to medical imaging, geophysics and signal processing in general. In this paper, we prove that the system of functions (so-called DOST basis) is indeed an orthonormal basis of $L^2([0,1])$, which is time–frequency localized, in the sense of Donoho–Stark Theorem (1989) [11]. Our approach provides a unified setting in which to study the Stockwell transform (associated with different admissible windows) and its orthogonal decomposition. Finally, we introduce a fast – $\mathcal{O}(N \log N)$ – algorithm to compute the Stockwell coefficients for an admissible window. Our algorithm extends the one proposed by Y. Wang and J. Orchard (2009) [33].

1. Introduction

Let f be a signal with finite energy, that is $f \in L^2(\mathbb{R})$, and let φ be a window in $L^2(\mathbb{R})$. Then, following M.W. Wong and H. Zhu [34], we define the Stockwell transform (S-transform) $\mathrm{S}_\varphi f$ as

$$(\mathrm{S}_\varphi f)(b,\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi i\, t\xi} f(t)\, |\xi|\, \overline{\varphi(\xi(t-b))}\, dt, \qquad b,\xi \in \mathbb{R}. \tag{1.1}$$

It is possible to rewrite the S-transform with respect to the Fourier transform of the analyzed signal:

$$(\mathrm{S}_\varphi f)(b,\xi) = \int_{\mathbb{R}} e^{2\pi i\, b\zeta}\, \widehat{f}(\zeta+\xi)\, \overline{\widehat{\varphi}\left(\frac{\zeta}{\xi}\right)}\, d\zeta, \qquad b,\xi \in \mathbb{R}, \quad \xi \neq 0, \tag{1.2}$$

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where $\widehat{f}$ is the Fourier transform of the signal f, given by

$$\widehat{f}(\xi) = (\mathrm{F} f)(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi \mathrm{i}\, t\xi} f(t) dt, \qquad \xi \in \mathbb{R}.$$

We fix the notation: we denote with $\check{f}$ or $\mathrm{F}^{-1} f$ the inverse Fourier transform of a signal f. $\mathbb{N} = \{0, 1, \ldots\}$ is the set of non-negative integers, $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots, \}$ is the set of integers.

The S-transform was initially defined by R.G. Stockwell, L. Mansinha and R.P. Lowe in [29] using a Gaussian window

$$g(t) = e^{-t^2/2}, \qquad t \in \mathbb{R}.$$

In this case,

$$(\mathrm{S}_g f)(b, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi \mathrm{i}\, t\xi} f(t)\, |\xi|\, e^{-(t-b)^2 \xi^2/2}\, dt, \qquad b, \xi \in \mathbb{R}, \tag{1.3}$$

which, in the alternative formulation, becomes

$$(\mathrm{S}_g f)(b, \xi) = \int_{\mathbb{R}} e^{2\pi \mathrm{i}\, \zeta b}\, \widehat{f}(\zeta + \xi)\, e^{-2\pi^2 \zeta^2/\xi^2}\, d\zeta, \qquad b, \xi \in \mathbb{R}, \quad \xi \neq 0. \tag{1.4}$$

The natural discretization of (1.4), introduced in [29], is given by

$$(\mathrm{S}_g f)(j, n) = \sum_{m=0}^{N-1} e^{2\pi \mathrm{i}\, mj/N}\, \widehat{f}(m + n)\, e^{-2\pi^2 m^2/n^2}, \tag{1.5}$$

where $j = 0, \ldots, N - 1$ and $n = 1, \ldots, N - 1$. For $n = 0$, it is set

$$(\mathrm{S}_g f)(j, 0) = \frac{1}{N} \sum_{k=0}^{N-1} f(k), \qquad j = 0, \ldots, N - 1.$$

In the literature, (1.5) is called redundant (discrete) Stockwell transform. Unfortunately, the redundant Stockwell transform has a high computational cost: $\mathcal{O}\left(N^2 \log N\right)$. To overcome this problem, R.G. Stockwell introduced in [27], without a mathematical proof, a basis for periodic signals with finite energy, *i.e.* $L^2([0, 1])$, given by

$$\bigcup_{p \in \mathbb{Z}} D_p = \bigcup_{p \in \mathbb{Z}} \{D_{p,\tau}\}_{\tau=0}^{\beta(p)-1}. \tag{1.6}$$

This basis, precisely defined in Section 3, is adapted to octave samples in the frequency domain. The decomposition of a periodic signal f in this basis is called in the literature the discrete orthonormal Stockwell transform (DOST). The related coefficients

$$f_{p,\tau} = (f, D_{p,\tau})_{L^2([0,1])},$$

are called DOST coefficients.