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Letter to the Editor

A strong restricted isometry property, with an application to phaseless compressed sensing ☆

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ABSTRACT

The many variants of the restricted isometry property (RIP) have proven to be crucial theoretical tools in the fields of compressed sensing and matrix completion. The study of extending compressed sensing to accommodate phaseless measurements naturally motivates a strong notion of restricted isometry property (SRIP), which we develop in this paper. We show that if $A \in \mathbb{R}^{m\times n}$ satisfies SRIP and phaseless measurements $|Ax_0| = b$ are observed about a k-sparse signal $x_0 \in \mathbb{R}^n$, then minimizing the ℓ_1 norm subject to $|Ax| = b$ recovers x_0 up to multiplication by a global sign. Moreover, we establish that the SRIP holds for the random Gaussian matrices typically used for standard compressed sensing, implying that phaseless compressed sensing is possible from $O(k\log(en/k))$ measurements with these matrices via ℓ_1 minimization over $|Ax| = b$. Our analysis also yields an erasure robust version of the Johnson–Lindenstrauss Lemma.

1. Introduction

The restricted isometry property (RIP), first introduced by Candès and Tao [5], is one of the most commonly used tools in the study of sparse/low rank signal recovery problem. The RIP also has some connections to the Johnson–Lindenstrauss Lemma and its study has lead to new results about the lemma [4,14]. The aim of this paper is to present a strong restricted isometry property which naturally occurs when considering phaseless compressed sensing.

Given a vector $x_0 \in \mathbb{R}^n$ and a collection of phaseless measurements $b_j = |\langle a_j, x_0\rangle|$, $j = 1, 2, \ldots, m$ where $a_j \in \mathbb{R}^n$, *phase retrieval* consists of recovering x_0 up to a global sign, which has attracted much attention in

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recent years (cf. [1,2,8,9]). It has been established that the PhaseLift algorithm allows for stable and efficient phase retrieval of an arbitrary signal $x_0 \in \mathbb{C}^n$ from $O(n)$ measurements via semi-definite programming [8].

In practice, signals of interest are often sparse in some basis and in particular this occurs in some regimes of X-ray crystallography. It is natural to exploit this sparsity structure to minimize the number of measurements needed for recovery since measurement acquisition is expensive and can destroy the sample at hand. We define *phaseless compressed sensing* (PCS) as the problem of recovering a sparse signal from few such phaseless measurements. It was shown in [15] that a k-sparse signal x_0 can be recovered from $O(k^2 \log n)$ phaseless measurements via convex programming. Surprisingly, and in contrast to the case of compressed sensing from linear measurements, it was also established in [15] that the natural information theoretic lower-bound of $O(k \log n)$ measurements cannot be achieved using a naive semi-definite programming relaxation.

Meanwhile, phaseless measurements are generically injective modulo phase over k-sparse signals as soon as the over-sampling factor is 2 [15,21]. Thus, the combinatorially hard problem of finding

$$\operatorname*{argmin}_{x \in \mathbb{R}^n} \{\|x\|_0 \ \text{ subject to } \ |\langle a_j, x\rangle| = |\langle a_j, x_0\rangle|, j = 1, \ldots, m\} \tag{1.1}$$

yields x_0 modulo phase for $m \geq 2k - 1$ and a_j generic (see [21]). Moreover, $O(k \log(en/k))$ random Gaussian phaseless measurements separate signals well [19]. While minimizing sparsity in the $O(k \log(en/k))$ measurement regime is not clearly amenable to efficient algorithmic recovery, numerical experiments show that using a convex relaxation of the ℓ_0 “norm” is often exact [18,20,22]. We study here such a relaxation:

$$\hat{x} := \operatorname*{argmin}_{x \in \mathbb{R}^n} \{\|x\|_1 \ \text{ subject to } \ |\langle a_j, x\rangle| = |\langle a_j, x_0\rangle|, j = 1, \ldots, m\}. \tag{1.2}$$

In this paper, we focus on the case where $x_0 \in \mathbb{R}^n$ is a k-sparse real signal and $A = [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m \times n}$. Particularly, we show that (1.1) is equivalent to (1.2) provided that the matrix A satisfies the strong restricted isometry property. We furthermore show that a random $m \times n$ Gaussian matrix with $m = O(k \log(en/k))$ satisfies the strong restricted isometry property of order k with high probability. And hence, the ℓ_1 relaxation is exact with high probability with $O(k \log(en/k))$ Gaussian measurements, just as in traditional compressed sensing. While the constraint set of our relaxation is non-convex, this formulation is more amenable to algorithmic recovery, and has been studied in [18,20,22] with demonstrated empirical success of the proposed projection algorithms. Finally, we would like to point out that all the results above are over $\mathbb{R}$. The extension of these results to hold over $\mathbb{C}$ cannot follow the same line of reasoning and is the subject of future work.

The paper is organized as follows. In Section 2, we introduce the definition of strong restricted isometry property (SRIP) and show that if a matrix A satisfies SRIP, then (1.1) is equivalent to (1.2). Then, we show that $m \times n$ random Gaussian matrices typically used for linear compressed sensing, also satisfy the SRIP with high probability, which establishes the exactness of the ℓ_1 relaxation for $m = O(k \log(en/k))$ phaseless measurements. Section 3 provides some technical necessities. We present a strong version of the concentration of measure inequality in Section 4 which plays an important role in proving the main results of Section 2. Using this inequality, we derive a strong J–L Lemma which states that we get a dimensionality reduction of a set of points in Euclidean space with some distance distortion such that erasing a positive fraction of the coordinates of the reduction maintains to a certain degree the approximate distance preserving J–L property. This can be interpreted as an erasure robust version of the J–L Lemma. We present the proofs of the main results in Section 5 and finally give some future research directions in Section 6.