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An algebraic perspective on multivariate tight wavelet frames. II

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A B S T R A C T

Continuing our recent work in [5] we study polynomial masks of multivariate tight wavelet frames from two additional and complementary points of view: convexity and system theory. We consider such polynomial masks that are derived by means of the unitary extension principle from a single polynomial. We show that the set of such polynomials is convex and reveal its extremal points as polynomials that satisfy the quadrature mirror filter condition. Multiplicative structure of this polynomial set allows us to improve the known upper bounds on the number of frame generators derived from box splines. Moreover, in the univariate and bivariate settings, the polynomial masks of a tight wavelet frame can be interpreted as the transfer function of a conservative multivariate linear system. Recent advances in system theory enable us to develop a more effective method for tight frame constructions. Employing an example by S.W. Drury, we show that for dimension greater than 2 such transfer function representations of the corresponding polynomial masks do not always exist. However, for all wavelet masks derived from multivariate polynomials with non-negative coefficients, we determine explicit transfer function representations. We illustrate our results with several examples.

1. Introduction

A tight wavelet frame of $L_2(\mathbb{R}^d)$ is determined, via Fourier transform, by a finite set of trigonometric polynomials $p, a_1, \dots, a_N$. The trigonometric polynomial p enters as the unique ingredient into the multiplicative identity

$$\hat{\phi}(M^T\theta) = p(z)\hat{\phi}(\theta), \quad \theta \in \mathbb{R}^d, \ z_j = e^{i\theta_j}, \tag{1}$$

where M is a $d \times d$ matrix with integer entries whose eigenvalues are greater than 1 in absolute value. The identity (1) is called the two-scale relation, as it defines a representation of ϕ in terms of shifts of scaled

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versions of ϕ, i.e.

$$\phi(x) = |\det M| \sum_{\alpha\in\mathbb{Z}^d} p(\alpha)\phi(Mx-\alpha), \quad x \in \mathbb{R}^d. \tag{2}$$

Here, $p(z) = \sum_{\alpha\in\mathbb{Z}^d} p(\alpha)z^\alpha$ has finitely many nonzero coefficients $p(\alpha)$ and $z^\alpha = z_1^{\alpha_1}\cdots z_d^{\alpha_d}$.

The translation group $G = 2\pi M^{-T}\mathbb{Z}^d/2\pi\mathbb{Z}^d$ plays a central role in the discussion of the two-scale relation. Clearly, G is a finite group of order $m = |\det M|$. Throughout this article we maintain the notation and terminology introduced in [5]. Our main object of study, as in the previous article [5], is the *mask* p, regarded as a Laurent polynomial or, equivalently, a trigonometric polynomial on the d-dimensional torus

$$\mathbb{T}^d = \{z = (z_1,\ldots,z_d) \in \mathbb{C}^d : |z_j| = 1 \text{ for } j = 1,\ldots,d\}.$$

An element of the group $\sigma = (\sigma_1,\ldots,\sigma_d) \in G$ acts on $p \in \mathbb{C}[\mathbb{T}^d]$ by

$$p^\sigma(z) := p\big(e^{-i\sigma_1}z_1,\ldots,e^{-i\sigma_d}z_d\big), \quad z \in \mathbb{T}^d.$$

The conditions

$$p^\sigma(1,1,\ldots,1) = \delta_{0,\sigma}, \quad \sigma \in G, \tag{3}$$

are called zero conditions or sum rules of order 1 in the literature, see [16] and references therein, and are important for the analysis of various properties of ϕ. Another important ingredient of the analysis is the fact that the support of ϕ is contained in the convex hull of $\{\alpha \in \mathbb{Z}^d : p(\alpha) \neq 0\}$.

We let $F_p = (p^\sigma)_{\sigma\in G}$, $F_{a_j} = (a_j^\sigma)_{\sigma\in G} : \mathbb{T}^d \to \mathbb{C}^m$ be column vectors. Then the identity

$$I_m - F_p(z)F_p(z)^* = \sum_{j=1}^N F_{a_j}(z)F_{a_j}(z)^* \tag{4}$$

is called the *Unitary Extension Principle* (UEP) in the seminal work on frames and shift-invariant spaces by Ron and Shen [22]. Here, $F_p(z)^* = \overline{F_p(z)}^T$ denotes complex conjugation and transposition. If the identities (3) and (4) are satisfied, then the functions

$$\psi_j(x) = |\det M| \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\phi(Mx-\alpha), \quad x \in \mathbb{R}^d, \tag{5}$$

are the generators of a tight wavelet frame; i.e. the family

$$X(\Psi) = \{m^{j/2}\psi_l\big(M^j\cdot -k\big) : 1 \le l \le N,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$$

defines a tight frame of $L_2(\mathbb{R}^d)$. Therefore, the UEP is the core of many constructions of tight wavelet frames, see e.g. [6,7,9,11,14,21,22,25].

The constraint

$$f = 1 - \sum_{\sigma\in G} p^{\sigma *}p^\sigma \ge 0 \tag{6}$$

is known in the literature as the sub-QMF condition on the trigonometric polynomial $p \in \mathbb{C}[\mathbb{T}^d]$. Due to $f = \det(I_m - F_pF_p^*)$, the condition in (6) is necessary for the existence of $a_1,\ldots,a_N$ that satisfy the UEP identities in (4).