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Continuous wavelet transforms on n-dimensional spheres

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ABSTRACT

In this paper, we are concerned with n-dimensional spherical wavelets derived from the theory of approximate identities. For nonzonal bilinear wavelets introduced by Ebert et al. in 2009 we prove isometry and Euclidean limit property. Further, we develop a theory of linear wavelets. In the end, we discuss the relationship to other wavelet constructions.

1. Introduction

Continuous wavelet transform based on singular integrals on the 2-sphere was introduced in [20] in 1996 (compare also [21] and [22]). The wavelets are zonal functions (they are in principle derivatives of kernels of singular integrals) such that the parameter space of the wavelet transform is $\mathbb{S}^2 \times \mathbb{R}_+$. Both linear and bilinear wavelet transforms are considered, where *linear* means that the decomposition is done by the wavelet, and no wavelet is needed for the reconstruction, whereas in the *bilinear* case one uses a wavelet both for the analysis and for the synthesis, compare also [22, Remark on p. 232].

A generalization to three and more dimensions was done by Bernstein et al. starting from 2009, cf. [7,9,8]. A special case of diffusive wavelets (i.e., satisfying an additional condition of diffusivity) is studied in Ebert's Ph.D. thesis [15].

In [9] a bilinear wavelet transform over $SO(3)$ is introduced with wavelets zonal with respect to an $SO(3)$ transform g_0. This idea is further developed in [16] to nonzonal bilinear wavelets over $SO(n)$. In this paper, we prove some properties of the nonzonal bilinear wavelet transform over $SO(n)$ and we introduce a

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nonzonal linear wavelet transform over $SO(n)$. Finally, we discuss the relationship between these and other wavelet constructions.

The paper is organized as follows. Section 2 contains basic information about analysis of functions on spheres. The notions of singular integrals and approximate identities are recapitulated and illustrated on some examples. In Subsection 2.3 we show that dilations via stereographic projections that were used in [1] create a kernel of an approximate identity. Section 3 digests and completes information about nonzonal and zonal bilinear wavelet transform introduced in [16], especially it is shown that spherical wavelets under some mild additional conditions have the Euclidean limit property, i.e., for small scales they behave like wavelets over Euclidean space. In Section 4 we introduce linear spherical wavelets and discuss their properties. Relationship between the presented constructions and other wavelets used for analysis of spherical functions is briefly discussed in Section 5.

2. Preliminaries

2.1. Functions on the sphere

By $\mathcal{S}^n$ we denote the n-dimensional unit sphere in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the rotation-invariant measure $d\sigma$ normalized such that

$$\Sigma_n = \int_{\mathcal{S}^n} d\sigma = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}.$$

The surface element $d\sigma$ is explicitly given by

$$d\sigma = \sin^{n-1}\theta_1 \, \sin^{n-2}\theta_2 \dots \sin\theta_{n-1} d\theta_1 \, d\theta_2 \dots d\theta_{n-1} d\varphi,$$

where $(\theta_1, \theta_2, \dots, \theta_{n-1}, \varphi) \in [0,\pi]^{n-1} \times [0, 2\pi)$ are spherical coordinates satisfying

$$\begin{aligned}
x_1 &= \cos\theta_1, \\
x_2 &= \sin\theta_1 \cos\theta_2, \\
x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
&\dots \\
x_{n-1} &= \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-2} \cos\theta_{n-1}, \\
x_n &= \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-2} \sin\theta_{n-1} \cos\varphi, \\
x_{n+1} &= \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-2} \sin\theta_{n-1} \sin\varphi.
\end{aligned}$$

$\langle x, y\rangle$ or $x \cdot y$ stands for the scalar product of vectors with origin in O and endpoint on the sphere. As long as it does not lead to misunderstandings, we identify these vectors with points on the sphere.

By $\mathcal{X}(\mathcal{S}^n)$ we denote the space $\mathcal{C}(\mathcal{S}^n)$ or $\mathcal{L}^p(\mathcal{S}^n)$, $1 \le p < \infty$, with norm given by

$$\|f\|_{\mathcal{C}(\mathcal{S}^n)} = \sup_{x\in\mathcal{S}^n} \bigl|f(x)\bigr|$$

or

$$\|f\|_{\mathcal{L}^p(\mathcal{S}^n)} = \left[\frac{1}{\Sigma_n} \int_{\mathcal{S}^n} \bigl|f(x)\bigr|^p \, d\sigma(x)\right]^{1/p},$$