Appl. Comput. Harmon. Anal. 39 (2015) 277–299

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Phase retrieval from coded diffraction patterns

Emmanuel J. Candès [a], Xiaodong Li [b], Mahdi Soltanolkotabi [c,*]

[a] Departments of Mathematics and of Statistics, Stanford University, Stanford, CA, United States
[b] Department of Statistics, The Wharton School, University of Pennsylvania, Philadelphia, PA, United States
[c] Department of Electrical Engineering, Stanford University, Stanford, CA, United States

ARTICLE INFO

Article history:
Received 20 November 2013
Received in revised form 15 September 2014
Accepted 23 September 2014
Available online 6 October 2014
Communicated by Radu Balan

Keywords:
Phase retrieval
Diffraction
Fourier transform
Convex optimization
Random matrix theory

ABSTRACT

This paper considers the question of recovering the phase of an object from intensity-only measurements, a problem which naturally appears in X-ray crystallography and related disciplines. We study a physically realistic setup where one can modulate the signal of interest and then collect the intensity of its diffraction pattern, each modulation thereby producing a sort of coded diffraction pattern. We show that *PhaseLift*, a recent convex programming technique, recovers the phase information exactly from a number of random modulations, which is polylogarithmic in the number of unknowns. Numerical experiments with noiseless and noisy data complement our theoretical analysis and illustrate our approach.

1. Introduction

1.1. The phase retrieval problem

In many areas of science and engineering, we only have access to magnitude measurements; for instance, it is far easier for detectors to record the modulus of the scattered radiation than to measure its phase. Imagine then that we have a discrete object $\boldsymbol{x} \in \mathbb{C}^n$, and that we would like to measure $\langle \boldsymbol{a}_k, \boldsymbol{x} \rangle$ for some sampling vectors $\boldsymbol{a}_k \in \mathbb{C}^n$ but only have access to phaseless measurements of the form

$$y_k = \left| \langle \boldsymbol{a}_k, \boldsymbol{x} \rangle \right|^2, \quad k = 1, \ldots, m. \tag{1.1}$$

The phase retrieval problem is that of recovering the missing phase of the data $\langle \boldsymbol{a}_k, \boldsymbol{x} \rangle$. Once this information is available, one can find the vector $\boldsymbol{x}$ by essentially solving a system of linear equations.

The quintessential phase retrieval problem, or phase problem for short, asks to recover a signal from the modulus of its Fourier transform. This comes from the fact that in coherent X-ray imaging, it follows from

* Corresponding author.

http://dx.doi.org/10.1016/j.acha.2014.09.004
1063-5203/

the Fraunhofer diffraction equation that the optical field at the detector is well approximated by the Fourier transform of the object of interest. Since photographic plates, CCDs and other light detectors can only measure light intensity, the problem is then to recover $\boldsymbol{x} = \{x[t]\}_{t=0}^{n-1} \in \mathbb{C}^n$ from measurements of the type

$$y_k = \left| \sum_{t=0}^{n-1} x[t] e^{-i2\pi\omega_k t} \right|^2, \quad \omega_k \in \Omega, \tag{1.2}$$

where Ω is a sampled set of frequencies in $[0, 1]$ (we stated the problem in one dimension to simplify matters). We thus recognize an instance of (1.1) in which the vectors $\boldsymbol{a}_k$ are sampled values of complex sinusoids. X-ray diffraction images are of this form, and as is well known, permitted the discovery of the double helix [55]. In addition to X-ray crystallography [33,39], the phase problem has numerous other applications in the imaging sciences such as diffraction and array imaging [18,24], optics [54], speckle imaging in astronomy [26], and microscopy [38]. Other areas where related problems appear include acoustics [12,8], blind channel estimation in wireless communications [4,45], interferometry [28], quantum mechanics [25,47] and quantum information [34].

1.2. Convex relaxation

Previous work [20,24] suggested to bring convex programming techniques to bear on the phase retrieval problem. Returning to the general formulation (1.1), the phase problem asks to recover $\boldsymbol{x} \in \mathbb{C}^n$ subject to data constraints of the form

$$\operatorname{tr}\left(\boldsymbol{a}_k \boldsymbol{a}_k^* \boldsymbol{x}\boldsymbol{x}^*\right) = y_k, \quad k = 1, \ldots, m,$$

where $\operatorname{tr}(\boldsymbol{X})$ is the trace of the matrix $\boldsymbol{X}$. The idea is then to lift the problem in higher dimensions: introducing the Hermitian matrix variable $\boldsymbol{X} \in \mathcal{S}^{n\times n}$, the phase problem is equivalent to finding $\boldsymbol{X}$ obeying

$$\boldsymbol{X} \succeq 0, \qquad \operatorname{rank}(\boldsymbol{X}) = 1, \qquad \operatorname{tr}\left(\boldsymbol{a}_k \boldsymbol{a}_k^* \boldsymbol{X}\right) = y_k \quad \text{for } k = 1, \ldots, m \tag{1.3}$$

where, here and below, $\boldsymbol{X} \succeq 0$ means that $\boldsymbol{X}$ is positive semidefinite. This problem is not tractable and, by dropping the rank constraint, is relaxed into

$$\begin{array}{ll} \text{minimize} & \operatorname{tr}(\boldsymbol{X}) \\ \text{subject to} & \boldsymbol{X} \succeq 0 \\ & \operatorname{tr}(\boldsymbol{a}_k \boldsymbol{a}_k^* \boldsymbol{X}) = y_k, \quad k = 1, \ldots, m. \end{array} \tag{1.4}$$

PhaseLift (1.4) is a semidefinite program (SDP). If its solution happens to have rank one and is equal to $\boldsymbol{x}\boldsymbol{x}^*$, then a simple factorization recovers $\boldsymbol{x}$ up to a global phase/sign.

We pause to emphasize that in different contexts, similar convex relaxations for optimizing quadratic objectives subject to quadratic constraints are known as Schor's semidefinite relaxations, see [41, Section 4.3] and [30] on the MAXCUT problem from graph theory for spectacular applications of these ideas. For related convex relaxations of quadratic problems, we refer the interested reader to the wonderful tutorial [52].

1.3. This paper

Numerical experiments [20] together with emerging theory suggest that the PhaseLift approach is surprisingly effective. On the theoretical side, starting with [23], a line of work established that if the sampling vectors $\boldsymbol{a}_k$ are sufficiently randomized, then the convex relaxation is provably exact. Assuming that the