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Letter to the Editor

Nonuniqueness of phase retrieval for three fractional Fourier transforms

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ABSTRACT

We prove that, regardless of the choice of the angles θ_1, θ_2, θ_3, three fractional Fourier transforms F_{θ_1}, F_{θ_2} and F_{θ_3} do not solve the phase retrieval problem. That is, there do not exist three angles θ_1, θ_2, θ_3 such that any signal $\psi \in L^2(\mathbb{R})$ could be determined up to a constant phase by knowing only the three intensities $|F_{\theta_1}\psi|^2$, $|F_{\theta_2}\psi|^2$ and $|F_{\theta_3}\psi|^2$. This provides a negative argument against a recent speculation by P. Jaming, who stated that three suitably chosen fractional Fourier transforms are good candidates for phase retrieval in infinite dimension. We recast the question in the language of quantum mechanics, where our result shows that any fixed triple of rotated quadrature observables Q_{θ_1}, Q_{θ_2} and Q_{θ_3} is not enough to determine all unknown pure quantum states. The sufficiency of four rotated quadrature observables, or equivalently fractional Fourier transforms, remains an open question.

1. Introduction

The problem of phase retrieval deals with reconstructing the phase of a complex signal from intensity measurements, that is, measurements which give as an outcome only the modulus of the signal. This problem is encountered in a wide variety of practical circumstances such as microscopy and crystallography [1]. In the context of quantum mechanics, it can be traced back to W. Pauli, who noted in a footnote in [2] that the question whether or not the position distribution $|\psi|^2$ and the momentum distribution $|\widehat{\psi}|^2$ uniquely determine the wave function ψ, "has still not been investigated in all its generality". It was soon realized that the distributions $|\psi|^2$ and $|\widehat{\psi}|^2$ do not determine the wave function up to a phase [3], and therefore one

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is led to search for a larger class of measurements that would be sufficient for the task at hand (see e.g. [4,5] for some more recent developments).

A natural direction for extending the original Pauli problem is given by the fractional Fourier transforms. The fractional Fourier transforms are a family of unitary operators F_θ, $\theta \in [0, 2\pi)$, on $L^2(\mathbb{R})$ which generalize the usual Fourier transform in such a way that (i) $F_0 = I$, (ii) $F_{\pi/2} = F$, the usual Fourier transform, (iii) $F_\pi = \Pi$, the parity operator, (iv) $F_{3\pi/2} = F^{-1}$ and (v) $F_{\theta_1} F_{\theta_2} = F_{\theta_1+\theta_2}$ where addition is understood *modulo* 2π (see [6,7]). The problem then is to determine if the knowledge of the intensities $|F_\theta \psi|^2$ for some suitable set of angles is sufficient for determining an arbitrary signal $\psi \in L^2(\mathbb{R})$ up to a constant phase. In the recent article [8], Jaming carried out this line of approach and obtained several interesting results in a number of cases where the signal is known to belong to some restricted class. In particular, in [8, Theorem 5.5] he proved that if ψ belongs to the dense subset of finite linear combinations of Hermite functions, then already two suitably chosen fractional Fourier transforms are sufficient. This immediately raises the question if three angles would be enough for an arbitrary signal. Indeed, Jaming himself suggests that “the fractional Fourier transform is a good candidate” for providing three unitary operators which would guarantee the uniqueness of phase retrieval.

It is the purpose of this Letter to show that regardless of the choice of the angles, three fractional Fourier transforms are not enough to ensure the uniqueness in the phase retrieval problem for arbitrary signals. We prove this by first formulating the question in the context of quantum mechanics, in which case knowledge of the modulus of the fractional Fourier transform corresponds to a measurement of a rotated quadrature observable (see formula (3) below). The problem is then turned into the analysis of the operator systems generated by sets of quadrature observables, rather than directly dealing with the corresponding Fourier operators. In this way, its solution is much simpler, as it essentially boils down to the analysis of symplectic 2×2-matrices. We then close this Letter by showing that our method no longer works for four angles, and therefore the exhaustive answer to the question regarding the minimal number of fractional Fourier transforms for unique phase retrieval remains an open question. These results should be compared to similar ones in the finite-dimensional setting, where it is known that uniqueness for the phase retrieval can be achieved with four unitary operators [8,9], and at least for sufficiently high dimensions this is the minimal number [10–12].

2. Quantum mechanical formulation of the problem

In quantum mechanics, the description of a physical system is based on a complex separable Hilbert space $\mathcal{H}$. We use the notation $\langle \cdot | \cdot \rangle$ for the inner product on $\mathcal{H}$ which, following the convention of the physics literature, we assume to be linear in the second argument.

Let $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ denote the Banach spaces of bounded and trace class operators on $\mathcal{H}$, respectively. The physical *states* of the system are represented by elements $\varrho \in \mathcal{T}(\mathcal{H})$ satisfying positivity $\varrho \geq 0$ and normalization $\mathrm{tr}[\varrho] = 1$. The states form a convex set whose extreme points, called the *pure states*, are precisely the one dimensional projections $|\varphi\rangle\langle\varphi| : \mathcal{H} \to \mathcal{H}$, $\|\varphi\| = 1$, defined via $|\varphi\rangle\langle\varphi|\psi = \langle\varphi|\psi\rangle\varphi$, with φ and ψ in $\mathcal{H}$. The *observables* are represented by normalized positive operator valued measures (POVMs) $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra of $\mathbb{R}$ [13,14]. More precisely, an observable is a map $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ which satisfies (i) positivity $\mathsf{E}(X) \geq 0$ for all $X \in \mathcal{B}(\mathbb{R})$, (ii) normalization $\mathsf{E}(\mathbb{R}) = I$, and (iii) σ-additivity $\mathrm{tr}[\varrho\mathsf{E}(\bigcup_j X_j)] = \sum_j \mathrm{tr}[\varrho\mathsf{E}(X_j)]$ for all states $\varrho \in \mathcal{T}(\mathcal{H})$ and all sequences $(X_j)_j$ of pairwise disjoint Borel sets. It follows that for any state ϱ, the map $\varrho^{\mathsf{E}} : \mathcal{B}(\mathbb{R}) \to [0, 1]$ defined via $\varrho^{\mathsf{E}}(X) = \mathrm{tr}[\varrho\mathsf{E}(X)]$ is a probability measure, and the number $\varrho^{\mathsf{E}}(X)$ is interpreted as the probability that the measurement of E gives an outcome from the set X, when the system is initially prepared in the state ϱ. In this article we are mainly interested in projection valued observables, that is, ones which satisfy $\mathsf{E}(X)^2 = \mathsf{E}(X)$. By the spectral theorem [15, X.4.11], these are in one-to-one correspondence with selfadjoint operators on $\mathcal{H}$.