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Uniqueness results in an extension of Pauli's phase retrieval problem

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A B S T R A C T

In this paper, we investigate an extension of Pauli's phase retrieval problem. The original problem asks whether a function u is uniquely determined by its modulus $|u|$ and the modulus of its Fourier transform $|\mathcal{F}u|$ up to a constant phase factor. Here we extend this problem by considering the uniqueness of the phase retrieval problem for the fractional Fourier transform (FrFT) of variable order. This problem occurs naturally in optics and quantum physics.
More precisely, we show that if u and v are such that fractional Fourier transforms of order α have same modulus $|\mathcal{F}_\alpha u| = |\mathcal{F}_\alpha v|$ for some set τ of α's, then v is equal to u up to a constant phase factor. The set τ depends on some extra assumptions either on u or on both u and v. Cases considered here are u, v of compact support, pulse trains, Hermite functions or linear combinations of translates and dilates of Gaussians. In this last case, the set τ may even be reduced to a single point (*i.e.* one fractional Fourier transform may suffice for uniqueness in the problem).

1. Introduction

Usually, when one measures a quantity, due to the nature of measurement equipment, noise, transmission in messy media... the phase of the quantity one wishes to know is lost. In mathematical terms, one wants to determine a quantity $\varphi(t)$ knowing only $|\varphi(t)|$ for all $t \in \mathbb{R}^d$. Stated as this, the problem has of course too many solutions to be useful and one tries to incorporate *a priori* knowledge on φ to decrease the under-determination. Problems of that kind are called *Phase Retrieval Problems* and arise in such diverse fields as microscopy (*see e.g.* [24,30,35,48,67,68]), holography [28,64], crystallography [50,60], neutron radiography [4], optical coherence tomography [61], optical design [27], radar signal processing [39], quantum mechanics [20,21,39,37,43,45] to name a few. We refer to the books [36,63], the review articles [40,50,29,42] for descriptions of various instances of this problem, some solutions to it (both theoretical and numerical) and for further references.

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The particular instance of the problem we are concerned with here deals with the so-called *Fractional Fourier Transform* (FrFT). Let us sketch a definition of this transform that is sufficient for the needs of the introduction (a precise definition follows in Section 3.3.1). First, for $u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we define the Fourier transform as

$$\mathcal{F}u(\xi) = \widehat{u}(\xi) = \int_{\mathbb{R}^d} u(t)e^{-2i\pi\langle t,\xi\rangle}\,\mathrm{d}t, \quad \xi \in \mathbb{R}^d$$

and then extend it to $L^2(\mathbb{R}^d)$ in the usual way. Here and throughout the paper $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are respectively the standard Euclidean norm on $\mathbb{R}^d$ and the corresponding scalar product. The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$. For $\alpha \in \mathbb{R} \setminus \pi\mathbb{Z}$, we define the fractional Fourier transform of order α via

$$\mathcal{F}_\alpha u(\xi) = c_\alpha e^{-i\pi|\xi|^2 \cot\alpha} \mathcal{F}\big[e^{-i\pi|\cdot|^2 \cot\alpha} u\big](\xi/\sin\alpha)$$

where c_α is a normalisation constant. We define $\mathcal{F}_0 u(\xi) = u(\xi)$, $\mathcal{F}_\pi u(\xi) = u(-\xi)$. Also note that, $\mathcal{F}_{\pi/2} = \mathcal{F}$, $\mathcal{F}_{-\pi/2} = \mathcal{F}^{-1}$ and that $\mathcal{F}_\alpha \mathcal{F}_\beta = \mathcal{F}_{\alpha+\beta}$.

This transform appears naturally in many instances including optics [56], quantum mechanics [45,52], signal processing [5,56]... We will detail below several instances where the fractional Fourier transform occurs and where one is further led to the question of recovery of a function $u \in L^2(\mathbb{R})$ from the phase-less measurements of several fractional Fourier transforms $\{|\mathcal{F}_\alpha u|\}_{\alpha\in\tau}$. More precisely, we deal with the following question:

Problem 1 *(Phase retrieval problem for the fractional Fourier transform). Let $u, v \in L^2(\mathbb{R}^d)$ and let $\tau \subset [0,\pi)$ be a set of indices (finite or not). Assume that*

$$|\mathcal{F}_\alpha v| = |\mathcal{F}_\alpha u| \quad \text{for every } \alpha \in \tau. \tag{1.1}$$

(i) *Does this imply that $v = cu$ for some constant $c \in \mathbb{C}$, $|c| = 1$?*
(ii) *Assume further that $u \in \mathcal{D}$ for some set $\mathcal{D} \subset L^2(\mathbb{R}^d)$. Does (1.1) imply that $v = cu$ for some constant $c \in \mathbb{C}$, $|c| = 1$?*
(iii) *Assume further that both $u, v \in \mathcal{D}$ for some set $\mathcal{D} \subset L^2(\mathbb{R}^d)$. Does (1.1) imply that $v = cu$ for some constant $c \in \mathbb{C}$, $|c| = 1$?*

In the first two cases we say that u is uniquely determined (up to constant multiples or up to a constant phase factor) from $\{|\mathcal{F}_\alpha u|, \alpha \in \tau\}$. In the last case we say that u is uniquely determined (up to a constant phase factor) from $\{|\mathcal{F}_\alpha u|, \alpha \in \tau\}$ within the class $\mathcal{D}$.

The most common phase retrieval problem is the case $\tau = \{\pi/2\}$ (*i.e.* when $\mathcal{F}_\alpha = \mathcal{F}$ is the usual Fourier transform) within the class $\mathcal{D}$ of compactly supported functions or distributions. Apart from this, the most famous problem of this sort is due to Pauli (*see e.g.* Reichenbach [59]) who asked whether $|u|$ and $|\mathcal{F}u|$ uniquely determine u up to constant phase factors *i.e.* here $\tau = \{0, \pi/2\}$. Several counter-examples to this question have been constructed (*see e.g* [21,37,39] and [20] for the state of the art on the problem). In order to construct those counter-examples, it was shown that u is not uniquely determined from $|u|$ and $|\mathcal{F}u|$ within the class $\{\sum_{\text{finite}} a_i\gamma(x-x_i)\}$ where γ is the standard Gaussian nor within the class $\{\sum_{\text{finite}} a_i\chi_{[0,1/2]}(x-i)\}$. We will show that for α well chosen, $|u|$ and $|\mathcal{F}_\alpha u|$ uniquely determine u up to constant multiples within these classes.

Note that Ron Wright (*see* [70]) conjectured that there is a unitary operator U on $L^2(\mathbb{R}^d)$ such that $|f|$, $|\hat{f}|$ and $|Uf|$ uniquely determine f up to a constant phase factor. Our results thus show that the fractional