Appl. Comput. Harmon. Anal. 37 (2014) 442–463

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Structure of nonstationary Gabor frames and their dual systems

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ARTICLE INFO

Article history:
Received 7 May 2013
Received in revised form 3 December 2013
Accepted 20 January 2014
Available online 23 January 2014
Communicated by Zuowei Shen

Keywords:
Time-frequency
Adaptivity
Gabor analysis
Frames
Duality condition

ABSTRACT

We investigate the structural properties of dual systems for nonstationary Gabor frames. In particular, we prove that some inverse nonstationary Gabor frame operators admit a Walnut-like representation, i.e. the operator acting on a function can be described by weighted translates of that function. In this case, which only occurs when compactly supported window functions are used, the canonical dual frame partially inherits the structure of the original frame, with differences that we describe in detail. Moreover, we determine a necessary and sufficient condition for a pair of nonstationary Gabor frames to form dual frames, valid under mild restrictions. This condition is then applied in a simple setup, to prove the existence of dual pairs of nonstationary Gabor systems with coarser frequency sampling than allowed by previous results [3]. We also explore a connection to recent work of Christensen, Kim and Kim on Gabor frames with compactly supported window function.

1. Introduction

This contribution investigates the properties of adaptive time-frequency systems that generalize classical Gabor systems. Although some of the presented results apply in a more general setting, the focus is on time-frequency systems with compactly supported generators.

Let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}^+$. The corresponding *Gabor system* [24,22] $\mathcal{G}(g,a,b)$ is the set of functions

$$g_{m,n}(t) = \mathbf{M}_{mb}\mathbf{T}_{na}g(t) = g(t-na)e^{2\pi imbt}, \quad \forall m,n \in \mathbb{Z} \tag{1}$$

with the translation and modulation operators, given by $\mathbf{T}_x g = g(\cdot - x)$ and $\mathbf{M}_x g = g(\cdot)e^{2\pi i x \cdot}$. The prototype function g is also called *window* or *generator function*. Of particular interest are systems that allow for stable, perfect reconstruction of any function $f \in L^2(\mathbb{R})$ from the system coefficients, given by inner products with the system elements. Such systems are generally called *frames* [20,7] or, when they are of the form $\mathcal{G}(g,a,b)$, *Gabor frames*. Gabor frames $\mathcal{G}(g,a,b)$ possess the remarkable property that the existence of a dual frame with the same structure, $\mathcal{G}(h,a,b)$ for some $h \in L^2(\mathbb{R})$, is guaranteed. This inheritance of structure from

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1063-5203/$ – see front matter
http://dx.doi.org/10.1016/j.acha.2014.01.004

the original frame by a dual frame does not hold for more general frames and is one important reason for the elegance of Gabor frames in both theory and application.

One of the early and most prevalent results in the field is the theory of painless non-orthogonal expansions [16], where the authors determine a simple sufficient condition for Gabor frames $\mathcal{G}(g, a, b)$ with compactly supported generator to constitute a frame. The condition guarantees that the frame operator is a multiplication operator and thus easily inverted. This setting is often referred to as the *painless case*, cf. Proposition 1.

In applications, frames generated from compactly supported window functions are of particular interest, because they allow for the most efficient computation of the frame coefficients and reconstruction. Compact support of the frame generators is also crucial for real-time implementation. Thus, the investigation of such frames beyond the painless case is an active field, see e.g. [6,8,13,11,31] and [10,9]. In the latter two articles, Christensen, Kim and Kim prove that for any Gabor frame with $\mathrm{supp}(g) \subseteq [1, 1]$, $a = 1$ and $b \in]1/2, 1[$, there exists a dual Gabor frame generated by a window supported on some compact set dependent only on the magnitude of b, cf. [10, Theorem 2.1, Lemma 3.2]. In fact, they show in [9] that the support condition in [10] can be further improved, for a large class of window functions g. Gröchenig and Stöckler recently proved the existence of dual frames with compactly supported, piecewise continuous generator for $\mathcal{G}(g, a, b)$ with g a totally positive function of finite type [25, Theorem 9]. They observe that the support size of this dual generator grows proportionally to the quotient $\frac{ab}{1-ab}$. The investigation of support properties for the dual frames of more general time-frequency systems is a central purpose of this contribution.

Although Gabor systems possess a number of useful mathematical properties that lead to a deep, yet accessible theory, the fixed time-frequency resolution and sampling strategy they provide is often debated as too restrictive for practical purposes. As a result, various generalizations have been proposed, to provide more flexible sampling strategies or varying window functions. Methods that allow for prefect reconstruction with flexible sampling and varying windows are scarce, however. One construction that unites these desirable properties are *nonstationary Gabor* (NSG) systems, first proposed by Jaillet [28]. While classical Gabor systems are constructed from regular translations and modulations, NSG systems are generated by a countable set of window functions and modulations thereof. Explicitly, we associate a sequence of pairs $\mathcal{G}(\mathbf{g}, \mathbf{b}) := (g_n, b_n)_{n\in\mathbb{Z}}$, $g_n \in L^2(\mathbb{R})$ and $b_n \in \mathbb{R}^+$, with the set of functions

$$g_{m,n}(t) = \mathbf{M}_{mb_n} g_n(t) = g_n(t)e^{2\pi i m b_n t}, \quad \text{for all } m, n \in \mathbb{Z}. \tag{2}$$

If $\mathcal{G}(\mathbf{g}, \mathbf{b})$ constitutes a frame, we call it a *nonstationary Gabor frame*. Note that an NSG system with $b_n = b$ and $g_n = T_{na} g$ for all $n \in \mathbb{Z}$ with $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}^+$ is a Gabor system.

Nonstationary Gabor frames combine the adaptivity of *local Fourier bases* [1,35] with the flexibility of redundant systems to provide a powerful framework for time-frequency representations. Much like Gabor frames give rise to *Wilson bases* [17,21,4,5], local Fourier bases can be constructed from NSG frames, although the more intricate properties of their relationship have yet to be investigated.

State of the art results on NSG frames are collected in [3], where an extension of the painless case to NSG systems is also given, see also Proposition 1.

The young theory of NSG frames beyond the painless case is currently being developed [18,19], while the painless construction is being used in realizing various time- or frequency-adaptive transforms [42,34,33,3,36,27].

Note that, in contrast to regular Gabor frames, the existence of a dual frame with the same structure, i.e. comprised of window functions h_n and modulation parameters b_n, is not guaranteed for general NSG frames. Indeed, one of the central results in this manuscript details the structure of the canonical dual system under certain restrictions. These restrictions, concerning the support and overlap of the window functions g_n and the modulation parameters b_n, guarantee compact support for the elements of the canonical dual frame and a certain modulation and phase shift structure, detailed in Section 5.