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A null space analysis of the ℓ_1-synthesis method in dictionary-based compressed sensing

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A B S T R A C T

An interesting topic in compressed sensing aims to recover signals with sparse representations in a dictionary. Recently the performance of the ℓ_1-analysis method has been a focus, while some fundamental problems for the ℓ_1-synthesis method are still unsolved. For example, what are the conditions for it to stably recover compressible signals under noise? Do coherent dictionaries allow the existence of sensing matrices that guarantee good performances of the ℓ_1-synthesis method? To answer these questions, we build up a framework for the ℓ_1-synthesis method. In particular, we propose a dictionary-based null space property (**D**-NSP) which, to the best of our knowledge, is the first sufficient and necessary condition for the success of ℓ_1-synthesis without measurement noise. With this new property, we show that when the dictionary **D** is full spark, it cannot be too coherent otherwise the ℓ_1-synthesis method fails for all sensing matrices. We also prove that in the real case, **D**-NSP is equivalent to the stability of ℓ_1-synthesis under noise.

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1. Introduction

Compressed sensing addresses the problem of recovering a sparse signal $\mathbf{z}_0 \in \mathbb{F}^d$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) from its undersampled and corrupted linear measurements $\mathbf{y} = \mathbf{A}\mathbf{z}_0 + \mathbf{w} \in \mathbb{F}^m$, where $\mathbf{w}$ is the noise vector such that $\|\mathbf{w}\|_2 \leqslant \epsilon$. The number of measurements m is usually much smaller than the ambient dimension d, which makes the problem ill-posed in general. A vector is said to be *s-sparse* if it has at most s nonzero entries. The sparsity of $\mathbf{z}_0$ makes the reconstruction possible. The following optimization algorithm, also known as Basis Pursuit, can reconstruct $\mathbf{z}_0$ efficiently from the perturbed observation $\mathbf{y}$ [5,15]:

$$\hat{\mathbf{z}} = \arg\min_{\mathbf{z}\in\mathbb{F}^d} \|\mathbf{z}\|_1, \quad \text{s.t. } \|\mathbf{y} - \mathbf{A}\mathbf{z}\|_2 \leqslant \epsilon. \tag{1}$$

A primary task of compressed sensing is to choose appropriate sensing matrix $\mathbf{A}$ in order to achieve good performance of (1). Candes and Tao proposed the *restricted isometry property* (*RIP*), and show that it

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provides stable reconstruction of approximately sparse signals via (1) [7]. Moreover, many random matrices satisfy RIP with high probability [6,25].

Another well-known condition on the measurement matrix is the null space property. A matrix $\mathbf{A}$ is said to have the *Null Space Property of order s* (s-NSP) if

$$\forall \mathbf{v} \in \ker \mathbf{A} \backslash \{0\},\ \forall |T| \leqslant s, \quad \|\mathbf{v}_T\|_1 < \|\mathbf{v}_{T^c}\|_1, \tag{2}$$

where $|T|$ is the cardinality for the index set $T \subset \{1, 2, \ldots, d\}$, T^c is its complementary index set and $\mathbf{v}_T$ is the restriction of $\mathbf{v}$ on T. NSP is known to characterize the exact reconstruction of all s-sparse vectors via (1) when there is no noise ($\epsilon = 0$) [13,16]. It has also been proven that the NSP matrices admit a similar stability result as RIP except that the constants may be larger [2,27].

In all of the above discussions, it is assumed that the signal $\mathbf{z}_0$ is sparse with respect to an orthonormal basis. A recent direction of interest in compressed sensing concerns problems where signals are sparse in an overcomplete dictionary $\mathbf{D}$ instead of a basis, see [24,4,21,2,11]. Here $\mathbf{D}$ is a $d \times n$ matrix with full column rank. We also call $\mathbf{D}$ a frame in the sense that the columns of $\mathbf{D}$ form a finite frame. *A finite frame for* $\mathbb{F}^d$ is a finite collection of vectors that span $\mathbb{F}^d$. We refer interested readers to [8] for a background on frame theory.

In this setting, the signal $\mathbf{z}_0 \in \mathbb{F}^d$ can be represented as $\mathbf{z}_0 = \mathbf{D}\mathbf{x}_0$, where $\mathbf{x}_0$ is an s-sparse vector in $\mathbb{F}^n$. We refer to such signals as *dictionary-sparse signals* or *frame-sparse signals*. When the dictionary $\mathbf{D}$ is specified, we also call them $\mathbf{D}$-*sparse signals*. We refer to the problem of recovering such $\mathbf{z}_0$ from the linear measurement $\mathbf{y} = \mathbf{A}\mathbf{z}_0$ as *dictionary-based compressed sensing*, and the ordinary compressed sensing problem as *basis-based compressed sensing*.

A natural way to obtain a good approximation $\hat{\mathbf{z}}$ of $\mathbf{z}_0$ is to use the following approach

$$(\mathrm{P}_{\mathbf{D}}) \quad \hat{\mathbf{x}} = \arg\min \|\mathbf{x}\|_1 \quad \text{s.t. } \mathbf{A}\mathbf{D}\mathbf{x} = \mathbf{y}, \tag{3}$$

$$\hat{\mathbf{z}} = \mathbf{D}\hat{\mathbf{x}}. \tag{4}$$

The above method is called the ℓ_1-synthesis or synthesis based method [21,24] due to the second synthesizing step. In the case when the measurements are perturbed, we naturally solve the following:

$$(\mathrm{P}_{\mathbf{D},\epsilon}) \quad \hat{\mathbf{x}} = \arg\min \|\mathbf{x}\|_1 \quad \text{s.t. } \|\mathbf{A}\mathbf{D}\mathbf{x} - \mathbf{y}\|_2 \leqslant \epsilon,$$
$$\hat{\mathbf{z}} = \mathbf{D}\hat{\mathbf{x}}.$$

The frame-based compressed sensing is motivated by the widespread use of overcomplete dictionaries and frames in signal processing and data analysis. Many signals naturally possess sparse frame coefficients, such as radar images (Gabor frames [23,17,26]), cartoon like images (curvelets [20]), images with directional features (shearlets [19]), etc. Other useful frames include wavelet frames [12] and harmonic frames. If the underlying frame is unknown but training data is available, the frame may also be constructed or approximated by learning. The greater flexibility and stability of frames make them preferable for practical purposes to achieve greater accuracy under imperfect measurements.

Despite the countless applications of frame-sparse signals, the compressed sensing literature is still lacking on this subject, especially on the issue whether the frame $\mathbf{D}$ can be allowed to be highly coherent or not. Coherence is a quantity that measures the correlation between frame vectors. When all the columns $\{\mathbf{d}_j\}$ of $\mathbf{D}$ are normalized, its *coherence* is defined as

$$\mu(\mathbf{D}) = \max_{i \neq j} \left| \langle \mathbf{d}_i, \mathbf{d}_j \rangle \right|.$$

A highly coherent $\mathbf{D}$ is a frame with big coherence.