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Letter to the Editor

The embedding dimension of Laplacian eigenfunction maps

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ABSTRACT

Any closed, connected Riemannian manifold M can be smoothly embedded by its Laplacian eigenfunction maps into $\mathbb{R}^m$ for some m. We call the smallest such m the maximal embedding dimension of M. We show that the maximal embedding dimension of M is bounded from above by a constant depending only on the dimension of M, a lower bound for injectivity radius, a lower bound for Ricci curvature, and a volume bound. We interpret this result for the case of surfaces isometrically immersed in $\mathbb{R}^3$, showing that the maximal embedding dimension only depends on bounds for the Gaussian curvature, mean curvature, and surface area. Furthermore, we consider the relevance of these results for shape registration.

1. Introduction

Let $M = (M, g)$ be a closed (compact, without boundary), connected Riemannian manifold; we assume both M and g are smooth. The Laplacian of M is a differential operator given by $\Delta := -\mathrm{div} \circ \mathrm{grad}$, where div and grad are the Riemannian divergence and gradient, respectively. Since M is compact and connected, Δ has a discrete spectrum $\{\lambda_j\}_{j\in\mathbb{N}}$, $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \uparrow \infty$. We may choose an orthonormal basis for $L^2(M)$ of eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ of Δ, where $\Delta\varphi_j = \lambda_j\varphi_j$, $\varphi_j \in C^\infty(M)$, $\varphi_0 \equiv V(M)^{-1/2}$. Here, $V(M)$ denotes the volume of M with respect to the canonical Riemannian measure $V = V_{(M,g)}$.

We consider maps of the form

$$\begin{aligned} \Phi^m : M &\longrightarrow \mathbb{R}^m \\ x &\longmapsto \{\varphi_j(x)\}_{1\leqslant j\leqslant m}. \end{aligned} \tag{1}$$

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If $\Phi^m : M \to \mathbb{R}^m$ happens to be a smooth embedding, then we call it an *m-dimensional eigenfunction embedding* of M. The smallest number m for which Φ^m is an embedding for *some* choice of basis $\{\varphi_j\}_{j\in\mathbb{N}}$ will herein be called the *embedding dimension* of M, and the smallest number m for which Φ^m is an embedding for *every* choice of basis $\{\varphi_j\}_{j\in\mathbb{N}}$ will be called the *maximal embedding dimension* of M. Our aim is to establish a (qualitative) bound for the maximal embedding dimension of a given Riemannian manifold in terms of basic geometric data.

That finite eigenfunction maps of the form (1) yield smooth embeddings for large enough m appears in a few papers in the spectral geometry literature. Abdallah [1] traces this fact back to Bérard [2]. To our knowledge, the latest embedding result is given in Theorem 1.3 in Abdallah [1], who shows that when $(M, g(t))$ is a family of Riemannian manifolds with $g(t)$ analytic in a neighborhood of $t = 0$, then there are $\epsilon > 0$, $m \in \mathbb{N}$, and eigenfunctions $\{\varphi_j(t)\}_{1\leqslant j\leqslant m}$ of $\Delta_{g(t)}$ such that

$$\begin{aligned} \big(M, g(t)\big) &\longrightarrow \mathbb{R}^m \\ x &\longmapsto \big\{\varphi_j(x;t)\big\}_{1\leqslant j\leqslant m} \end{aligned} \tag{2}$$

is an embedding for all $t \in (-\epsilon, \epsilon)$. The proof does not suggest how topology and geometry determine the embedding dimension, however.

Jones, Maggioni, and Schul [3,4] have studied local properties of eigenfunction maps, and their results are essential to the proof of our main result. In particular, they show that at $z \in M$, for an appropriate choice of weights $a_1, \dots, a_n \in \mathbb{R}$ and eigenfunctions $\varphi_{j_1}, \dots, \varphi_{j_n}$, one has a coordinate chart (U, Φ_a) around $z \in M$, where $\Phi_a(x) := (a_1\varphi_{j_1}(x), \dots, a_n\varphi_{j_n}(x))$, satisfying $\|\Phi_a(x) - \Phi_a(y)\|_{\mathbb{R}^n} \sim d_M(x, y)$ for all $x, y \in U$. A more explicit statement of this result is given below.

Minor variants of such eigenfunction maps have been used in a variety of contexts. For example, spectral embeddings

$$\begin{aligned} M &\longrightarrow \ell^2 \\ x &\longmapsto \big\{e^{-\lambda_j t/2}\varphi_j(x)\big\}_{j\in\mathbb{N}} \quad (t > 0) \end{aligned} \tag{3}$$

have been used to embed closed Riemannian manifolds into the Hilbert space ℓ^2 (i.e. square summable sequences with the usual inner product) in Bérard, Besson, and Gallot [5,6]; Fukaya [7]; Kasue and Kumura, e.g. [8,9]; Kasue, Kumura, and Ogura [10]; Kasue, e.g. [11,12]; and Abdallah [1].

Relatives of the eigenfunction maps, or a discrete counterpart, have been studied for data parametrization and dimensionality reduction, e.g. [13–18]; for shape distances, e.g. [19–22]; and for shape registration, e.g. [23–29]. In particular, in the data analysis community, (1) is known as the *eigenmap* [13], (3) is known as the *diffusion map* [15,16], and $x \mapsto \{\lambda_j^{-1/2}\varphi_j(x)\}$ is known as the *global point signature* [18]. These maps are all equivalent up to an invertible linear transformation. Hence, any embedding result applies to all of them. For an overview of spectral geometry in shape and data analysis, we refer the reader to Mémoli [22].

There seem to be no rules for choosing the number of eigenfunctions to use for a given application. While not all applications require an (injective) embedding of data, many eigenfunction-based shape registration methods do, e.g. [24–29], as we explain in Section 1.1 below. In the discrete setting one can write an algorithm to determine the smallest m for which $\Phi^m : M \to \mathbb{R}^m$ is an embedding, although such an approach may become computationally intensive. For example, if M is represented as a polyhedral surface, one may write an algorithm to check for self-intersections of polygon faces in the image $\Phi^m : M \to \mathbb{R}^m$. The fail-proof approach is to use all eigenfunctions, in which case one is assured an embedding. This approach is mentioned for point cloud data in Coifman and Lafon [16]. Specifically, they bound the maximal embedding dimension from above by the size of the full point sample. This becomes computationally demanding, however, especially in applications where one must solve an optimization problem over all eigenspaces, e.g. [21,24,25,28], as we discuss in Section 1.1. Under the assumption that the shape or data is a sample drawn