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Letter to the Editor

Phase retrieval for sparse signals

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A B S T R A C T

The aim of this paper is to build up the theoretical framework for the recovery of sparse signals from the magnitude of the measurements. We first investigate the minimal number of measurements for the success of the recovery of sparse signals from the magnitude of samples. We completely settle the minimality question for the real case and give a bound for the complex case. We then study the recovery performance of the ℓ_1 minimization for the sparse phase retrieval problem. In particular, we present the null space property which, to our knowledge, is the first sufficient and necessary condition for the success of ℓ_1 minimization for k-sparse phase retrieval.

1. Introduction

The theory of compressive sensing has generated enormous interest in recent years. The goal of compressive sensing is to recover a sparse signal from its linear measurements, where the number of measurements is much smaller than the dimension of the signal, see e.g. [4–6,12]. The aim of this paper is to study the problem of compressive sensing without the phase information. In this problem the goal is to recover a sparse signal from the magnitude of its linear samples.

Recovering a signal from the magnitude of its linear samples, commonly known as *phase retrieval* or *phaseless reconstruction*, has gained considerable attention in recent years [1,2,7,8]. It has important application in X-ray imaging, crystallography, electron microscopy, coherence theory and other applications. In many applications the signals to be reconstructed are sparse. Thus it is natural to extend compressive sensing to the phase retrieval problem.

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We first introduce the notation and briefly describe the mathematical background of the problem. Let $\mathcal{F} = \{f_1, f_2, \dots, f_m\}$ be a set of vectors in $\mathbb{H}^d$ where $\mathbb{H}$ is either $\mathbb{R}$ or $\mathbb{C}$. Assume that $x \in \mathbb{H}^d$ such that $b_j = |\langle x, f_j\rangle|$. The phase retrieval problem asks whether we can reconstruct x from $\{b_j\}_{j=1}^m$. Obviously, if $y = cx$ where $|c| = 1$ then $|\langle y, f_j\rangle| = |\langle x, f_j\rangle|$. Thus the best phase retrieval can do is to reconstruct x up to a unimodular constant.

Consider the equivalence relation $\sim$ on $\mathbf{H} := \mathbb{H}^d$: $x \sim y$ if and only if there is a constant $c \in \mathbb{H}$ with $|c| = 1$ such that $x = cy$. Let $\widetilde{\mathbf{H}} := \mathbf{H}/\sim$. We shall use $\tilde{x}$ to denote the equivalent class containing x. For a given $\mathcal{F}$ in $\mathbf{H}$ define the map $\mathbf{M}_{\mathcal{F}} : \widetilde{\mathbf{H}} \longrightarrow \mathbb{R}_+^m$ by

$$\mathbf{M}_{\mathcal{F}}(\tilde{x}) = \left[\left|\langle \tilde{x}, f_1\rangle\right|^2, \dots, \left|\langle \tilde{x}, f_m\rangle\right|^2\right]^\top. \tag{1.1}$$

The phase retrieval problem asks whether an $\tilde{x} \in \widetilde{\mathbf{H}}$ is uniquely determined by $\mathbf{M}_{\mathcal{F}}(\tilde{x})$, i.e. $\tilde{x}$ is recoverable from $\mathbf{M}_{\mathcal{F}}(\tilde{x})$. We say that a set of vectors $\mathcal{F}$ has the *phase retrieval property*, or is *phase retrievable*, if $\mathbf{M}_{\mathcal{F}}$ is injective on $\widetilde{\mathbf{H}} = \mathbb{H}^d/\sim$.

It is known that in the real case $\mathbb{H} = \mathbb{R}$ the set $\mathcal{F}$ needs to have at least $m \geq 2d - 1$ vectors to have the phase retrieval property; furthermore a generic set of $m \geq 2d-1$ elements in $\mathbb{R}^d$ will have the phase retrieval property, (cf. Balan, Casazza and Edidin [1]). In the complex case $\mathbb{H} = \mathbb{C}$ the same question remains open, and is perhaps the most prominent open problem in phase retrieval. It is known that $m \geq 4d - 2$ generic vectors in $\mathbb{C}^d$ have the phase retrieval property [1]. The result is improved to $m \geq 4d - 4$ in [10]. The $m = 4d - 4$ vectors having the phase retrieval property are also constructed in [3]. The current conjecture is that phase retrieval property in $\mathbb{C}^d$ can only hold when $m \geq 4d - 4$.

The aforementioned results concern the general phase retrieval problem in $\mathbb{H}^d$. In many applications, however, the signal x is often sparse with $\|x\|_0 = k \ll d$.

We use the standard notation $\mathbb{H}_k^d$ to denote the subset of $\mathbb{H}^d$ whose elements x have $\|x\|_0 \leq k$. Let $\widetilde{\mathbf{H}}_k$ denote $\mathbb{H}_k^d/\sim$. A set $\mathcal{F}$ of vectors in $\mathbb{H}^d$ is said to have the *k-sparse phase retrieval property*, or is *k-sparse phase retrievable*, if any $\tilde{x} \in \widetilde{\mathbf{H}}_k$ is uniquely determined by $\mathbf{M}_{\mathcal{F}}(\tilde{x})$. In other words, the map $\mathbf{M}_{\mathcal{F}}$ is injective on $\widetilde{\mathbf{H}}_k$. One may naturally ask: *How many vectors does $\mathcal{F}$ need to have so that $\mathcal{F}$ is k-sparse phase retrievable?*

The best current results on the k-sparse phase retrieval property are proved by Li and Voroninski [16], which state that k-sparse phase retrieval property can be achieved by having $m \geq 4k$ and $m \geq 8k$ vectors for the real and complex case, respectively (see also [18]).

In Section 2, we prove sharper results for a set of vectors $\mathcal{F}$ to have the k-sparse phase retrieval property. In the real case $\mathbb{H} = \mathbb{R}$ we obtain a sharp result. We show that for any $k < d$ the set $\mathcal{F}$ must have at least $m \geq 2k$ elements to be k-sparse phase retrievable. Furthermore, any $m \geq 2k$ generic vectors will be k-sparse phase retrievable. In the complex case $\mathbb{H} = \mathbb{C}$ we proved that any $m \geq 4k - 2$ generic vectors have the k-sparse phase retrieval property. We conjecture that this bound is also sharp, namely for $k < d$ a set $\mathcal{F}$ in $\mathbb{C}^d$ needs at least $4k - 2$ vectors to have the k-sparse phase retrieval property.

A foundation of compressive sensing is built on the fact that the recovery of a sparse signal from a system of under-determined linear equations is equivalent to finding the extremal value of ℓ_1 minimization under certain conditions. The ℓ_1 minimization is extended to the phase retrieval in [17] and one also develops many algorithms to compute it (see [20,22]). However, there have been few theoretical results on the recovery performance of ℓ_1 minimization for sparse phase retrieval. In Section 3, we present the null space property, which, to our knowledge, is the first sufficient and necessary condition for the success of ℓ_1 minimization for k-sparse phase retrieval. If we take $k = d$, the null space property is reduced to a condition of the set of vectors $\mathcal{F}$ under which $\mathbf{M}_{\mathcal{F}}$ is injective on $\mathbb{C}^d/\sim$ and we present it in Section 4.