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Frame properties of shifts of prolate spheroidal wave functions

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ABSTRACT

We provide conditions on a shift parameter and number of basic prolate spheroidal wave functions with a fixed bandwidth and time concentrated to a fixed duration such that the shifts of the basic prolates form a frame or a Riesz basis for the Paley–Wiener space consisting of all square integrable functions with the given bandlimit.

1. Introduction

A considerable literature has been developed on the subject of principal and finitely generated shift invariant spaces $V(\Psi)=\overline{\mathrm{span}}\{\psi_n(\cdot-k):n=1,\ldots,N;\,k\in\mathbb{Z}\}$, including structural results [5,20,3] and more recent results addressing spaces with further invariance properties [1,11,26]. An equally substantial literature has been developed on the subject of Gabor systems $\mathcal{G}(g_1,\ldots,g_N;\alpha,\beta)$ generated by time–frequency shifts $\mathrm{e}^{2\pi\mathrm{i}\ell\beta t}\,g_n(t-\alpha k)$ of a single or finite collection of generators, e.g., [7,8,4,10] with more recent applications outlined in [2,19]. A primary question is what properties of the generators g_n and shift parameters α,β are consistent with the Gabor system forming a frame or Riesz basis for $L^2(\mathbb{R})$. We study here very specific shift invariant systems with properties that are in a sense intermediate to structural properties of finitely generated shift invariant spaces on the one hand and of Gabor systems on the other. Specifically, we are interested in frame and Riesz basis properties of systems generated by shifts of certain *prolate spheroidal wave functions* (prolates, for short). These are bandlimited functions that are the most concentrated to a fixed time interval in L^2-norm. They are eigenfunctions of the operator that first truncates to a time interval

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then truncates to a frequency interval, and they are ordered by decreasing magnitude of the eigenvalue. The famous "$2\Omega T$" theorem states that, asymptotically, the number of independent unit norm eigenfunctions φ_n having eigenvalues λ_n close to one is the product of the bandwidth and the duration—the length of the time-concentration interval.

Denote by φ_n the nth prolate in this ordering, $n = 0, 1, 2, \ldots$, bandlimited to $[-\Omega, \Omega]$ and time-concentrated in the interval $[-1, 1]$ with associated eigenvalue λ_n. Fix $\alpha > 0$. We first consider frame properties of families of shifts $\{\sqrt{\lambda_n}\,\varphi_n(\cdot - 2k\alpha) : n = 0, 1, 2, \ldots, k \in \mathbb{Z}\}$ of $\sqrt{\lambda_n}$-normalized prolates. The factor *two* is used here to emphasize that we are shifting by multiples of the duration. The prolates φ_n form an orthonormal basis for the Paley–Wiener space $\mathrm{PW}_{2\Omega}$ of L^2-functions bandlimited to $[-\Omega, \Omega]$ as well as an orthogonal basis for $L^2[-1, 1]$, so it is mildly surprising that a collection of integer shifts of $\sqrt{\lambda_n}$-normalized prolates might form a frame for the Paley–Wiener space. In fact, Theorems 2 and 4 show that they form a tight frame in certain cases and at least a near-tight frame in others.

When considering shifts of the first N prolates $\varphi_0, \ldots, \varphi_{N-1}$, it is possible to obtain frames for the Paley–Wiener space with or without normalizing by $\sqrt{\lambda_n}$. The first main result in this regard, Theorem 10, essentially states that the family forms a frame for $\mathrm{PW}_{2\Omega}$ if $N \geq 2\Omega\alpha$, but that the family is incomplete in $\mathrm{PW}_{2\Omega}$ if $N < 2\Omega\alpha$. However, the frame bounds are less concretely quantified in this case. The proof of Theorem 10 illustrates why the family $\{\varphi_n(\cdot - \alpha k)\}_{n=0,\, k\in\mathbb{Z}}^{N-1}$ is redundant or overcomplete if $N > 2\Omega\alpha$. When $N = 2\Omega\alpha \in \mathbb{N}$ it is possible that the family $\{\varphi_n(\cdot - \alpha k)\}_{n=0,\, k\in\mathbb{Z}}^{N-1}$ actually forms a Riesz basis for $\mathrm{PW}_{2\Omega}$ and the second main result, Theorem 12, shows that this is indeed the case. Notationally we have absorbed the duration into the shift factor α in Theorems 10 and 12, considering shifts $\varphi(\cdot - \alpha k)$ rather than $\varphi(\cdot - 2\alpha k)$ as in the renormalized frames, in order to minimize notational burden in Sections 4 and 5.

When the prolate shifts $\{\varphi_n(t - \alpha k)\}_{n=0,\, k\in\mathbb{Z}}^{N-1}$ form a frame or Riesz basis for $\mathrm{PW}_{2\Omega}$, the time–frequency shifts $\{\mathrm{e}^{4\pi i\ell\Omega t}\varphi_n(t - \alpha k)\}_{n=0,\, k,\ell\in\mathbb{Z}}^{N-1}$ form a corresponding frame or Riesz basis for $L^2(\mathbb{R})$ with the same bounds simply because for $\ell \neq \ell'$, $\langle \mathrm{e}^{4\pi i\ell\Omega t}\varphi_n(t - \alpha k), \mathrm{e}^{4\pi i\ell'\Omega t}\varphi_n(t - \alpha k')\rangle = 0$ since the modulated prolates are frequency-supported on disjoint intervals. This fact does not contradict the Balian–Low theorem, e.g., [8] because $t\varphi_n(t) \notin L^2(\mathbb{R})$, e.g., [22, Eq. 1.10]. Analogues of Gabor frames generated by prolates were studied in the context of joint time–frequency cutoffs by Dörfler and Romero [6].

The remainder of the paper is organized as follows. In Section 2 we review necessary background properties of prolate spheroidal wave functions. Section 3 establishes frame properties of the families $\mathcal{F}_\alpha = \{\sqrt{\lambda_n}\,\varphi_n(\cdot - 2\alpha\ell) : n \geq 0, \ell \in \mathbb{Z}\}$, including Theorems 2 and 4, which provide explicit frame bounds for the Paley–Wiener space $\mathrm{PW}_{2\Omega}$. Section 4 shows that the unrenormalized families $\{\varphi_n(\cdot - \alpha\ell) : n = 0, \ldots, N-1, \ell \in \mathbb{Z}\}$ of shifts of the first N prolates form frames for $\mathrm{PW}_{2\Omega}$ under the condition that there is at least one prolate shift per unit time–bandwidth (Theorem 10), and in Section 5 it is shown that they form a Riesz basis for $\mathrm{PW}_{2\Omega}$ if there is precisely one prolate shift per unit time–bandwidth (Theorem 12).

2. Background on prolate spheroidal wave functions and frames

Much of the mathematical foundation of time and band limiting was laid out in a series of papers written by combinations of Landau, Slepian and Pollak [24,14,15,21,23] appearing in the *Bell System Technical Journal* in the early 1960s. For $T > 0$, the time-limiting operator Q_T corresponds to multiplying $f \in L^2(\mathbb{R})$ by $\mathbb{1}_{[-T,T]}$, the characteristic function of the interval $[-T, T]$. Let P_Ω denote the bandlimiting operator $P_\Omega = \mathcal{F}^{-1}Q_{\Omega/2}\mathcal{F}$ with $(\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-2\pi it\xi}\,\mathrm{d}t$ denoting the Fourier transform. The duration–bandwidth product is $2\Omega T$. The operator $P_\Omega Q_T$ is compact and self-adjoint on the Hilbert space $\mathrm{PW}_\Omega = P_\Omega(L^2(\mathbb{R}))$. The operators $Q = Q_1$ and $P_{c/\pi}$ commute with the differential operator

$$\mathcal{P}_c = \frac{\mathrm{d}}{\mathrm{d}t}\big(t^2 - 1\big)\frac{\mathrm{d}}{\mathrm{d}t} + c^2t^2 \tag{1}$$