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Left-invariant evolutions of wavelet transforms on the similitude group

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ABSTRACT

Enhancement of multiple-scale elongated structures in noisy image data is relevant for many biomedical applications but commonly used PDE-based enhancement techniques often fail at crossings in an image. To get an overview of how an image is composed of local multiple-scale elongated structures we construct a continuous wavelet transform on the similitude group, $SIM(2)$. Our unitary transform maps the space of images onto a reproducing kernel space defined on $SIM(2)$, allowing us to robustly relate Euclidean (and scaling) invariant operators on images to left-invariant operators on the corresponding continuous wavelet transform. Rather than often used wavelet (soft-)thresholding techniques, we employ the group structure in the wavelet domain to arrive at left-invariant evolutions and flows (diffusion), for contextual crossing preserving enhancement of multiple scale elongated structures in noisy images. We present experiments that display benefits of our work compared to recent PDE techniques acting directly on the images and to our previous work on left-invariant diffusions on Coherent state transforms defined on Euclidean motion group.

1. Introduction

Elongated structures in the human body such as fibres and blood vessels often require analysis for diagnostic purposes. A wide variety of medical imaging techniques such as magnetic resonance imaging (MRI), microscopy, X-ray fluoroscopy, fundus imaging etc. exist to achieve this. Many (bio)medical questions related to such images require detection and tracking of the elongated structures present therein. Due to the desire to reduce acquisition time and radiation dosage the acquired medical images are often noisy, of low contrast and suffer from occlusions and incomplete data. Furthermore multiple-scale elongated structures exhibit

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crossings and bifurcations which is a notorious problem in (medical) imaging. Hence crossing-preserving enhancement of these structures is an important preprocessing step for subsequent detection.

In recent years PDE based techniques have gained popularity in the field of image processing. Due to well posed mathematical results these techniques lend themselves to stable algorithms and also allow mathematical and geometrical interpretation of classical methods such as Gaussian and morphological filtering, dilation or erosion etc. on $\mathbb{R}^d$. These techniques typically regard the original image, $f \in \mathbb{R}^2 \to \mathbb{R}$, as an initial state of a parabolic (diffusion like) evolution process yielding filtered versions, $u_f : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$. Here u_f is called the scale space representation of image f. The domain of u_f is scale space $\mathbb{R}^2 \times \mathbb{R}^+$. A typical scale space evolution is of the form

$$\begin{cases} \partial_s u_f(\mathbf{x}, s) = \nabla_{\mathbf{x}} \cdot (C(u_f(\cdot, s))(\mathbf{x}) \nabla_{\mathbf{x}} u_f)(\mathbf{x}, s) \\ u_f(\mathbf{x}, 0) = f(\mathbf{x}), \end{cases} \tag{1}$$

where $C(u_f(\cdot, s))(\mathbf{x})$ models the diffusivity depending on the differential structure at $(\mathbf{x}, s, u_f(\mathbf{x}, s)) \in \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}$. For $C = 1$, Eq. (1) is the usual linear heat equation. The corresponding evolution is known in image processing as a Gaussian Scale Space [1–4]. In their seminal paper [5], Perona and Malik proposed nonlinear filters to bridge scale space and restoration ideas. Based on the observation that diffusion should not occur when the (local) gradient value is large (to avoid blurring the edges), they pointed out that nonlinear adaptive isotropic diffusion is achieved by replacing $C = 1$ by $C(u_f(\cdot, s))(\mathbf{x}) = c(\|\nabla_{\mathbf{x}} u_f(\mathbf{x}, s)\|)$, where $c : \mathbb{R}^+ \to \mathbb{R}^+$ is some smooth strictly decreasing positive function vanishing at infinity. An improvement of the Perona–Malik scheme is the "coherence-enhancing diffusion" (CED) introduced by Weickert [6] which additionally uses the direction of the gradient $\nabla_{\mathbf{x}} u_f$ leading to diffusion constant c being replaced by a nonlinear matrix.

However these methods often fail in image analysis applications with crossing or bifurcating curves as the direction of gradient at these structures is ill-defined, see [7] for more details. Scharr et al. in [8] present techniques which effectively deal with the particular case of X-junctions by relying on the 2-nd order jet of Gaussian derivatives in the image domain. Passing through higher order jets of Gaussian derivatives and induced Euclidean invariant differential operators does not allow one to generically deal with complex crossings and/or bifurcating structures. Instead we need gauge frames in higher dimensional Lie groups to deal with this issue. According to the terminology used in [9, Section 3.3.3] a gauge frame is a local coordinate system aligned/gauged with locally present (elongated) structures in an image. Differentiating w.r.t. such coordinates provides intrinsically natural derivatives as opposed to differentiating w.r.t. (artificially imposed) global coordinates. At salient locations in the image, where multiple scale elongated structures cross, one needs multiple gauge frames. Therefore instead of gauge frames per position, $x \in \mathbb{R}^2$ in a (grey-scale) image $f : \mathbb{R}^2 \to \mathbb{R}$, we attach gauge frames to each Lie group element,

$$g = (x, t) \in G = \mathbb{R}^2 \rtimes T.$$

in a Coherent state (CS) transform $\mathcal{W}_\psi f : G \to \mathbb{C}$ of an image $f : \mathbb{R}^2 \to \mathbb{R}$. In this article we mainly consider $(G = SE(2), T = SO(2))$ and $(G = SIM(2), T = \mathbb{R}^+ \times SO(2))$, where (multiple scale) elongated structures are manifestly disentangled via the transform, allowing for a crossing preserving flow (steered by gauge frames). In medical image processing $\mathcal{W}_\psi f$ for $G = SE(2)$ is also referred to as an orientation score as it provides a score of how an image is decomposed out of local (possibly crossing) orientations.

1.1. Why extend to the $SIM(2)$ group?

In this paper we wish to extend the aforementioned framework to the case of the similitude group (group of planar translations, rotations and scaling), for the following reasons: