Appl. Comput. Harmon. Anal. 39 (2015) 138–160

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Intrinsic modeling of stochastic dynamical systems using empirical geometry

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A R T I C L E I N F O

Article history:
Received 16 September 2013
Received in revised form 18 August 2014
Accepted 31 August 2014
Available online 6 September 2014
Communicated by Radu Balan

Keywords:
Intrinsic model
Nonlinear inverse problem
Differential geometry
Information geometry
Non-parametric estimation
Nonlinear dynamical systems
Manifold learning
Graph-based method

A B S T R A C T

In a broad range of natural and real-world dynamical systems, measured signals are controlled by underlying processes or drivers. As a result, these signals exhibit highly redundant representations, while their temporal evolution can often be compactly described by dynamical processes on a low-dimensional manifold. In this paper, we propose a graph-based method for revealing the low-dimensional manifold and inferring the processes. This method provides intrinsic models for measured signals, which are noise resilient and invariant under different random measurements and instrumental modalities. Such intrinsic models may enable mathematical calibration of complex measurements and build an empirical geometry driven by the observations, which is especially suitable for applications without a priori knowledge of models and solutions. We exploit the temporal dynamics and natural small perturbations of the signals to explore the local tangent spaces of the low-dimensional manifold of empirical probability densities. This information is used to define an intrinsic Riemannian metric, which in turn gives rise to the construction of a graph that represents the desired low-dimensional manifold. Such a construction is equivalent to an inverse problem, which is formulated as a nonlinear differential equation and is solved empirically through eigenvectors of an appropriate Laplace operator. We examine our method on two nonlinear filtering applications: a nonlinear and non-Gaussian tracking problem as well as a non-stationary hidden Markov chain scheme. The experimental results demonstrate the power of our theory by extracting the underlying processes, which were measured through different nonlinear instrumental conditions, in an entirely data-driven nonparametric way.

1. Introduction

Due to natural constraints, in a broad range of real-world applications, the accessible (high-dimensional) data exhibit typical structure and often lie on a (low-dimensional) manifold. In recent years, this observation gave rise to the development of manifold learning methods, which aim at finding parameterizations of the underlying low-dimensional structures of given data sets [1–5]. A similar observation applies to dynamical

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http://dx.doi.org/10.1016/j.acha.2014.08.006

systems: the measured output signal of the system is often controlled by few underlying drivers, whose temporal evolution can be compactly described by dynamical processes on a low-dimensional manifold [6–10]. This belief is naturally encoded in the standard state-space formalism used to describe dynamical systems: given signal measurements $\mathbf{z}_t$ in time t, we are interested in estimating the associated system state $\boldsymbol{\theta}_t$. We remark that the focus of this paper will be on the state estimation problem, which is typically extended to filtering, forecasting and prediction, as well as noise suppression problems by attaching statistical models in a Bayesian manner.

To support the analysis of time series and dynamical systems, the standard geometric setting of manifold learning needs to be extended. First, the mapping between the measured signal and the underlying processes is often stochastic and contains measurement noise. As a result, repeated measurements of the same phenomenon yield different measurement realizations. Furthermore, the measurements may be acquired using different instruments or sensors. Each set of related measurements of the same phenomenon will then have a different manifold, depending on the instrument and the specific realization. Thus, to provide meaningful information on the true state of the system, the geometric parameterization should be invariant to the measurement and instrumental modalities. Second, the dynamics of the signals carry essential information and should be encoded in the analysis results. Third, the ability to sequentially analyze streaming data is an important aspect of dynamical systems and signal processing. When a stream of new incoming signal samples becomes available, the manifold model needs to be efficiently extended.

In [11], Singer and Coifman addressed the problem in which the desired "interesting" data are accessible via an unknown nonlinear measurement function. To provide a model for the desired data, rather than the accessible measurements, the Mahalanobis distance was used to invert the measurement function locally, assuming that the function that maps the data into a set of measurements is deterministic and stably invertible on its range. As a result, their approach provides a model for the desired manifold, whereas classic manifold learning methods provide a model for the manifold of the observations.

In this paper, we extend [11] and propose a graph-based method for the parameterization of the underlying processes controlling stochastic dynamical systems, which we refer to as empirical intrinsic geometry (EIG). The primary focus of the paper is on the construction of an intrinsic Riemannian distance metric between measurement samples that exhibits the desired invariance to the measurement modality and noise. The construction of the metric is carried out in two scales of short-time windowing. In a micro-scale, local probability densities of the measurements are estimated in windows and viewed as descriptors, or features, where the sampling rate of the measurements is assumed to be sufficiently high to allow for an accurate densities estimation. We will motivate this particular choice of features by showing that any stationary measurement noise in the signal domain is translated to a linear operation in the probability densities domain. In a macro-scale, we exploit the temporal dynamics and natural small perturbations of the underlying process to explore the local tangent spaces to the manifold by estimating the covariances of the probability densities in time windows. Here we further assume that the signal is pseudo-stationary, i.e., its probability density is slowly changing with time. Based on the estimates of the local probability densities and their covariances, we compute the Mahalanobis distance [11]. Since the Mahalanobis distance is invariant to linear transformations, and since any measurement noise is translated to a linear transformation in the domain of probability densities, the constructed distance metric is invariant to moderate noise.

Despite drawing most of our attention in the present work, the Mahalanobis distance is an intermediate analysis result, since it provides merely a *local* Euclidean structure. Nevertheless, the availability of such an intrinsic distance metric enables us to construct diffusion geometry [5] via a graph that represents the intrinsic manifold of underlying processes. This construction is shown to be equivalent to an inverse problem, which is solved empirically through eigenvectors of an appropriate Laplace operator. Specifically, the eigenvectors provide an embedding (or a parametrization) of the underlying processes on the intrinsic manifold. We remark that the construction of the graph is implemented using a reference set of measurements