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Letter to the Editor

Stable recovery of analysis based approaches ☆

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A B S T R A C T

The theory of compressed sensing shows that it is highly possible to recover a sparse signal from few measurements. Due to its wide applications, compressed sensing has drawn attention of many researchers from the fields of signal and image processing, applied mathematics, and statistics. In this paper we are interested in signals which are sparse under redundant tight frames. Some sufficient conditions are provided to guarantee the stable recovery via solving analysis based approaches. Compared with the previous work [12,16], our sufficient conditions are weaker and the estimations of l_2 bound only depend on the measurement matrix.

1. Introduction

In compressed sensing, a vector $\boldsymbol{x} \in \mathbb{R}^n$ is called *k-sparse* if the number of its nonzero entries is at most $k(\ll n)$. If a vector can be approximated well by sparse vectors, it is called a *compressible* vector. Suppose that the measurements of a sparse vector $\boldsymbol{x}$ are given by

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{w},$$

where $\boldsymbol{w} \in \mathbb{R}^m$ is the noise and A is an $m \times n$ measurement matrix with $m \leqslant n$. The sparse vector $\boldsymbol{x}$ can be recovered via solving convex minimization problems, provided that the measurement matrix A satisfies some conditions such as mutual incoherence property [9] or restricted isometry property [5].

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This paper mainly considers the case that $\boldsymbol{x}$ is not sparse itself but is compressible with respect to some given tight frame D. If the l_2 norm of the noise $\boldsymbol{w}$ is bounded by ε, then the recovery problem can be formulated as the constrained analysis based approach:

$$\min_{\tilde{\boldsymbol{x}}\in\mathbb{R}^n} \|D\tilde{\boldsymbol{x}}\|_1 \quad \text{subject to} \quad \|A\tilde{\boldsymbol{x}}-\boldsymbol{y}\|_2 \leqslant \varepsilon. \tag{1.1}$$

The constrained analysis based approach is said to produce a *stable* estimation $\hat{\boldsymbol{x}}$ to the true vector $\boldsymbol{x}$ if

$$\|\hat{\boldsymbol{x}}-\boldsymbol{x}\|_2 \leqslant C_0\left(\varepsilon + \frac{\|D\boldsymbol{x}-(D\boldsymbol{x})_k\|_1}{\sqrt{k}}\right), \tag{1.2}$$

where $(D\boldsymbol{x})_k$ denotes the vector consisting of the largest k coefficients of $D\boldsymbol{x}$ in magnitude and C_0 is a constant. As a special case, if the coefficient vector $D\boldsymbol{x}$ is at most k-sparse and there is no noise in the measurements, then $\boldsymbol{x}$ is exactly recovered. A widely used condition that guarantees the stable signal recovery is the restricted isometry property adapted to D [3]. Let D^* be the complex conjugate of the transpose of a $d\times n$ matrix D. Since D is often real-valued, D^* in this paper is simply the transpose of D. Then D is a tight frame for $\mathbb{R}^n$ if and only if $D^*D = I_n$. The measurement matrix A satisfies the *restricted isometry property adapted to D* (D-RIP) if there is a constant $0<\delta_k<1$ such that

$$(1-\delta_k)\|\boldsymbol{v}\|_2^2 \leqslant \|A\boldsymbol{v}\|_2^2 \leqslant (1+\delta_k)\|\boldsymbol{v}\|_2^2 \tag{1.3}$$

holds for all $\boldsymbol{v}\in\Sigma_k := \{D^*\boldsymbol{c} : \boldsymbol{c}\in\mathbb{R}^d,\ \|\boldsymbol{c}\|_0 \leqslant k\}$. The smallest constant δ_k satisfying (1.3) is called the D-RIP constant. When D is the identity matrix, the constant δ_k is called the *restricted isometry property* (RIP) constant [5]. Therefore, the D-RIP is a natural extension of the RIP. Almost all random matrices, such as Gaussian, subgaussian, Bernoulli random matrices and subsampled Fourier matrices, satisfying the RIP also satisfy the D-RIP, up to multiplication by a random sign matrix (in the case of the random DFT, for example) [3]. It was first proved in [3] that the condition $\delta_{2k} < 0.08$ guarantees the stable recovery of the l_1 minimization model in (1.1) and the constant C_0 in (1.2) depends on the D-RIP constant δ_{2k}. This sufficient condition for stable recovery was later improved in [13,14].

Instead of solving (1.1) directly, many algorithms were proposed to solve the following unconstrained analysis based approach:

$$\min_{\tilde{\boldsymbol{x}}\in\mathbb{R}^n} \lambda\|D\tilde{\boldsymbol{x}}\|_1 + \frac{1}{2}\|A\tilde{\boldsymbol{x}}-\boldsymbol{y}\|_2^2. \tag{1.4}$$

See [10] for a recent overview on the analysis based approaches (1.4). Under some assumptions on the matrix A and the tight frame D, the relation between (1.1) and (1.4) has been discussed in [18]. Generally, the minimization of (1.1) and the minimization of (1.4) are not exactly the same. To the best of our knowledge, using D-RIP to guarantee the stable recovery, the condition on the D-RIP constant δ_{2k} for solving (1.4) is stronger than that for solving (1.1). Under the condition that $\|DA^*\boldsymbol{w}\|_\infty \leqslant \frac{\lambda}{2}$ and $\delta_{2k} < 0.0833$, the minimization solution $\hat{\boldsymbol{x}}$ to (1.4) satisfies

$$\|\hat{\boldsymbol{x}}-\boldsymbol{x}\|_2 \leqslant C_1\sqrt{k}\lambda + C_2\frac{\|D\boldsymbol{x}-(D\boldsymbol{x})_k\|_1}{\sqrt{k}}, \tag{1.5}$$

where the constant C_2 depends on the D-RIP constant while the constant C_1 depends on both the D-RIP constant and $\|DD^*\|_{1,1}$ [12]. Here $\|DD^*\|_{1,1}$ stands for the operator norm of DD^* acting on l_1, that is,

$$\|DD^*\|_{1,1} := \sup\big\{\|DD^*\boldsymbol{c}\|_1 : \boldsymbol{c}\in\mathbb{R}^d,\ \|\boldsymbol{c}\|_1 \leqslant 1\big\}.$$