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On minimal trajectories for mobile sampling of bandlimited fields

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A B S T R A C T

We study the design of sampling trajectories for stable sampling and the reconstruction of bandlimited spatial fields using mobile sensors. The spectrum is assumed to be a symmetric convex set. As a performance metric we use the path density of the set of sampling trajectories that is defined as the total distance traveled by the moving sensors per unit spatial volume of the spatial region being monitored. Focusing first on parallel lines, we identify the set of parallel lines with minimal path density that contains a set of stable sampling for fields bandlimited to a known set. We then show that the problem becomes ill-posed when the optimization is performed over all trajectories by demonstrating a feasible trajectory set with arbitrarily low path density. However, the problem becomes well-posed if we explicitly specify the stability margins. We demonstrate this by obtaining a non-trivial lower bound on the path density of an arbitrary set of trajectories that contain a sampling set with explicitly specified stability bounds.

1. Introduction

The reconstruction of a function from given measurements is a fundamental task in data processing and occupies numerous directions of research in mathematics and engineering. A typical problem requires the reconstruction or approximation of a physical field from pointwise measurements. A field may be a distribution of temperatures or water pollution or a solution to a diffusion equation, in mathematical terminology a field is simply a smooth function of several variables. The standard assumption on the smoothness is that the field is bandlimited to a compact spectrum. If the spectrum is a fundamental

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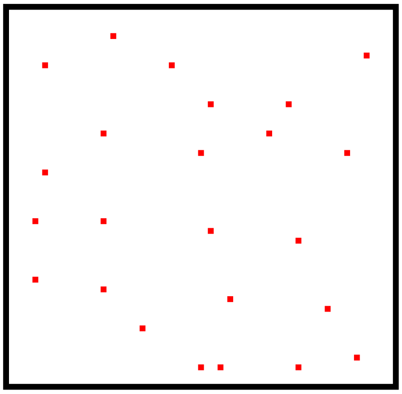
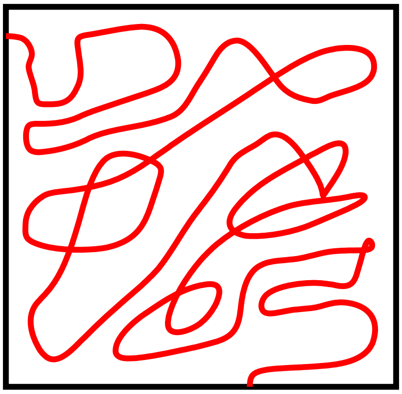


(a) Static sampling on points



(b) Mobile sampling on a curve

Fig. 1. Two approaches for sampling a field in $\mathbb{R}^2$.

domain of a lattice in $\mathbb{R}^d$ or a symmetric convex polygon in $\mathbb{R}^2$, then there exist precise reconstruction formulas from sufficiently many samples in analogy to the Shannon–Whittaker–Kotelnikov sampling theorem [15,21].

Let

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(r) e^{-2\pi \mathrm{i} \langle \omega, r \rangle} dr, \quad \omega \in \mathbb{R}^d, \tag{1}$$

be the Fourier transform[1] of $f \in L^1(\mathbb{R}^d)$ or $f \in L^2(\mathbb{R}^d)$, where i denotes the imaginary unit and $\langle \omega, r \rangle$ denotes the scalar product between vectors ω and r in $\mathbb{R}^d$. We say that f is bandlimited to the closed set $\Omega \subset \mathbb{R}^d$, if its Fourier transform $\hat{f}$ is supported on Ω. In this case we write

$$\mathcal{B}_\Omega := \left\{ f \in L^2\left(\mathbb{R}^d\right) : \widehat{f}(\omega) = 0 \text{ for almost every } \omega \notin \Omega \right\} \tag{2}$$

for the space of fields with finite energy bandlimited to the spectrum Ω. In the context of field estimation we always assume that the spectrum is a compact, symmetric, convex set.

The classical theory of sampling and reconstructing of such high-dimensional bandlimited fields dates back to Petersen and Middleton [21] in signal analysis and to Beurling [5] in harmonic analysis. Both identified conditions for reconstructing such fields from their point measurements in $\mathbb{R}^d$. Further research on non-uniform sampling generated more results on conditions for perfect reconstruction from samples taken at non-uniformly distributed spatial locations. See [9,10] and the survey [1]. Previous work deals primarily with the problem of reconstructing the field from measurements taken by a collection of static sensors distributed in space, like that shown in Fig. 1(a). In this case the performance metric for quantifying the efficiency of a sampling scheme is the spatial density of samples. This is the average number of sensors per unit volume required for the stable sampling of the monitored region.

In this paper we investigate a different method for the acquisition of the samples, which we call *mobile sampling*. The samples are taken by a mobile sensor that moves along a continuous path, as is shown in Fig. 1(b). In such a case it is often relatively inexpensive to increase the spatial sampling rate along the sensor's path while the main cost of the sampling scheme comes from the total distance that needs to be traveled by the moving sensor. Hence it is reasonable to assume that the sensor can record the field values at an arbitrarily high but finite resolution on its path.

[1] Note that in [23] and [26] the Fourier transform was defined without the 2π in the exponent.