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Letter to the Editor

Uniqueness of Gabor series

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ABSTRACT

We prove that any complete and minimal Gabor system of Gaussians is a Markushevich basis in $L^2(\mathbb{R})$.

1. Introduction

Let $\Lambda \subset \mathbb{R}^2$ be a sequence of distinct points. With each such sequence we associate Gabor system

$$\mathcal{G}_\Lambda := \{e^{2\pi i y t} e^{-\pi(t-x)^2}\}_{(x,y)\in\Lambda}. \tag{1.1}$$

Function $e^{2\pi i y t} e^{-\pi(t-x)^2}$ can be viewed as the time–frequency shift of the Gaussian $e^{-\pi t^2}$ in the phase space. It is well known that system $\mathcal{G}_\Lambda$ cannot be a Riesz basis in $L^2(\mathbb{R})$ (see e.g. [9]). On the other hand, there exist a lot of *complete and minimal* systems $\mathcal{G}_\Lambda$. A canonical example is the lattice without one point, $\Lambda := \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$. However, the generating sets Λ can be very far from any lattice. For example, in [1] it was shown that there exists $\Lambda \subset \mathbb{R}\times\{0\}\cup\{0\}\times\mathbb{R}$ such that $\mathcal{G}_\Lambda$ is complete and minimal in $L^2(\mathbb{R})$.

If $\mathcal{G}_\Lambda$ is complete and minimal, then there exists the unique biorthogonal system $\{g_{(x,y)}\}_{(x,y)\in\Lambda}$. So, for any $f \in L^2(\mathbb{R})$ we may write the formal Fourier series with respect to the system $\mathcal{G}_\Lambda$

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$$f \sim \sum_{(x,y)\in\Lambda} (f, g_{(x,y)})_{L^2(\mathbb{R})} e^{2\pi i y t} e^{-\pi(t-x)^2}. \tag{1.2}$$

If $\Lambda = \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$, then it is known that there exists a linear summation method for the series (1.2) (e.g. one can use methods from [8]). In [8] this was proved for certain sequences similar to lattices. The main point of the present note is to show that *any* series (1.2) defines an element f uniquely.

Theorem 1.1. *Let $\mathcal{G}_\Lambda$ be a complete and minimal system in $L^2(\mathbb{R})$. Then the biorthogonal system $\{g_{(x,y)}\}_{(x,y)\in\Lambda}$ is complete. So, any function $f \in L^2(\mathbb{R})$ is uniquely determined by the coefficients $(f, g_{(x,y)})$.*

This property is by no means automatic for an arbitrary system of vectors. Indeed, if $\{e_n\}_{n=1}^\infty$ is an orthonormal basis in a separable Hilbert space, then $\{e_1 + e_n\}_{n=2}^\infty$ is a complete and minimal system but its biorthogonal $\{e_n\}_{n=2}^\infty$ is not complete. A complete and minimal system in a Hilbert space with complete biorthogonal system is called *Markushevich basis*.

Theorem 1.1 is analogous to Young's theorem [11] for systems of complex exponentials $\{e^{i\lambda_n t}\}$ in L^2 of an interval. However, the structure of complete and minimal systems for Gabor systems is more puzzling than for the systems of exponentials on an interval. For example, if Λ generates a complete and minimal system of exponentials in $L^2(-\pi,\pi)$, then the upper density of Λ ($= \limsup_{r\to\infty} \#(\Lambda\cap\{|\lambda|<r\})(2r)^{-1}$) is equal to 1; see Theorem 1 in Lecture 17 of [7]. On the other hand, if $\mathcal{G}_\Lambda$ is a complete and minimal Gabor system, then the upper density of Λ ($= \limsup_{r\to\infty} \#(\Lambda\cap\{x^2+y^2\le r^2\})(\pi r^2)^{-1}$) can vary from $2/\pi$ to 1; see Theorem 1 in [1]. If, in addition, Λ is a regular distributed set, then the upper density have to be from $2/\pi$ to 1; see Theorem 2 in [1].

Note that for some systems of special functions (associated to some canonical system of differential equations) in L^2 of an interval completeness of the biorthogonal system may fail (even with infinite defect); see [2, Proposition 3.4].

In the next section we transfer our problem to the Fock space of entire functions. The last section is devoted to the proof of our result.

Notations. Throughout this paper the notation $U(x) \lesssim V(x)$ means that there is a constant C such that $U(x) \le CV(x)$ holds for all x in the set in question, $U, V \ge 0$. We write $U(x) \asymp V(x)$ if both $U(x) \lesssim V(x)$ and $V(x) \lesssim U(x)$.

2. Reduction to a Fock space problem

Let

$$\mathcal{F} := \{F \text{ is entire and } \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) < \infty\};$$

here dm denotes the planar Lebesgue measure. It is well known that the following Bargmann transform

$$\mathcal{B}f(z) := 2^{1/4} e^{-i\pi x y} e^{\frac{\pi}{2}|z|^2} \int_{\mathbb{R}} f(t) e^{2\pi i y t} e^{-\pi(t-x)^2} dt$$
$$= 2^{1/4} \int_{\mathbb{R}} f(t) e^{-\pi t^2} e^{2\pi t z} e^{-\frac{\pi}{2} z^2} dt, \quad z = x + iy,$$

is a unitary map between $L^2(\mathbb{R})$ and the Fock space $\mathcal{F}$; see [5,6] for the details.

Moreover, the time–frequency shift of the Gaussian is mapped to the normalized reproducing kernel of $\mathcal{F}$