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Letter to the Editor

Sparse disjointed recovery from noninflating measurements

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A B S T R A C T

We investigate the minimal number of linear measurements needed to recover sparse disjointed vectors robustly in the presence of measurement error. First, we analyze an iterative hard thresholding algorithm relying on a dynamic program computing sparse disjointed projections to upper-bound the order of the minimal number of measurements. Next, we show that this order cannot be reduced by any robust algorithm handling noninflating measurements. As a consequence, we conclude that there is no benefit in knowing the simultaneity of sparsity and disjointedness over knowing only one of these structures.

1. Introduction and main result

In this note, we examine the recovery of sparse disjointed vectors $\mathbf{x} \in \mathbb{C}^N$ from linear measurements $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{C}^m$ with $m \ll N$. We recall that a vector $\mathbf{x} \in \mathbb{C}^N$ is said to be s-sparse if it has no more than s nonzero entries, i.e., if $\mathrm{card}(\mathrm{supp}(\mathbf{x})) \le s$, where $\mathrm{supp}(\mathbf{x}) := \{i \in [\![1 : N]\!] : x_i \neq 0\}$. It is said to be d-disjointed if there are always at least d zero entries between two nonzero entries, i.e., if $|j - i| > d$ for all distinct $i, j \in \mathrm{supp}(\mathbf{x})$. We investigate here vectors that are simultaneously s-sparse and d-disjointed. This investigation was prompted by grid discretizations in MIMO radar problems [9]: the nonzero entries represent the positions of airplanes in an observation frame, so it is natural to assume that their number is low and that they are not too close to one another. Sparse disjointed vectors also serve as a pertinent model for neural spike trains, see [8] which already established recovery results akin to those presented in Section 3. In this note, however, we emphasize the question of the *minimal* number of measurements

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needed for robust uniform recovery of sparse disjointed vectors. We provide a complete answer with regard to noninflating measurements relative to this model (see Section 4 for the explanation of this terminology). As a reminder, the uniform recovery of s-sparse vectors is achievable from

$$m \asymp m_{\mathsf{spa}} := s \ln\left(e\frac{N}{s}\right) \tag{1}$$

random linear measurements. It can be carried out efficiently using convex optimization or iterative greedy algorithms. The recovery is robust with respect to measurement error and stable with respect to sparsity defect. The number of measurements in (1) is optimal when stability is required. As for the uniform recovery of d-disjointed vectors, it is achievable from

$$m \asymp m_{\mathsf{dis}} := \frac{N}{d} \tag{2}$$

deterministic Fourier measurements and it can be carried out efficiently using convex optimization (see [4, Corollary 1.4]). The number of measurements (2) is easily seen to be optimal, even without requiring stability. Concerning simultaneously sparse and disjointed vectors, our main result is informally stated below.

Theorem 1. *The minimal number of noninflating measurements needed to achieve robust uniform recovery of s-sparse d-disjointed vectors is of the order of*

$$m_{\mathsf{spa\&dis}} := s \ln\left(e\frac{N - d(s-1)}{s}\right). \tag{3}$$

The significance of this result lies in its interpretation: for $m_{\mathsf{spa\&dis}}$ to be of smaller order than m_{spa}, we need $t := (N - d(s-1))/s \leq N/(2s)$; but then $d = (N - st)/(s-1) \geq (N - N/2)/(s-1) \geq N/(2s)$, i.e., $N/d \leq 2s$, which implies that m_{dis} is of smaller order than $m_{\mathsf{spa\&dis}}$. In short, we arrive at

$$m_{\mathsf{spa\&dis}} \asymp \min\{m_{\mathsf{spa}}, m_{\mathsf{dis}}\}. \tag{4}$$

Expressed differently, there is no benefit in knowing the simultaneity of sparsity and disjointness as far as the number of noninflating measurements is concerned. This echoes the message of [10], which showed that vectors possessing certain structures simultaneously require at least as many Gaussian random measurements for their recovery via combined convex relaxations as what could have been achieved via the convex relaxation associated to one of the structures. Our result is narrower since it focuses on a particular simultaneity of structures, but no limitation is placed on the nature of the recovery algorithm and the measurements are only assumed to be noninflating instead of Gaussian. Note that restricting to ℓ_1-minimization and Gaussian measurements would have been irrelevant here, because even nonuniform recovery, i.e., the recovery of a single sparse vector—a fortiori of a disjointed one—already requires a number of measurements of order at least m_{spa}, as inferred from known results on phase transition (see [5] for the original arguments and [1] for recent arguments).

The rest of this note is organized as follows. In Section 2, we discuss basic facts about sparse disjointed vectors. In particular, we reveal how projections onto the set of sparse disjointed vectors can be computed by dynamic programming. The ability to compute these projections would allow for the modification of virtually all sparse recovery iterative greedy algorithms to fit the sparse disjointed framework, but we focus only on iterative hard thresholding (IHT)—arguably the simplest of these algorithms—in Section 3. There, we give a short justification that robust uniform recovery can be carried out efficiently based on random measurements (which are noninflating) provided their number has order at least $m_{\mathsf{spa\&dis}}$. Finally, Section 4