Appl. Comput. Harmon. Anal. 37 (2014) 347–359

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Letter to the Editor

Periodic wavelet frames and time–frequency localization

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ARTICLE INFO

Article history:
Received 28 August 2013
Received in revised form 22 February 2014
Accepted 27 February 2014
Available online 6 March 2014
Communicated by Qingtang Jiang

Keywords:
Periodic wavelet
Scaling function
Parseval frame
Tight frame
Uncertainty principle
Poisson summation formula
Localization

ABSTRACT

A family of Parseval periodic wavelet frames is constructed. The family has optimal time–frequency localization (in the sense of the Breitenberger uncertainty constant) with respect to a family parameter and it has the best currently known localization with respect to a multiresolution analysis parameter.

1. Introduction

In recent years the wavelet theory of periodic functions has been continuously refined. First, periodic wavelets were generated by periodization of wavelet functions on the real line (see, for example, [6]). A wider and more natural approach providing a flexibility on a theoretical front and in applications is to study periodic wavelets directly using a periodic analog of a multiresolution analysis (MRA). The concept of periodic MRA is introduced and discussed in [14,18–21,26,27,29]. In [9], a unitary extension principle (UEP) for constructing Parseval wavelet frames is rewritten for periodic functions (see Theorem 1). The approach is developed further in [8].

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[1] Supported by RSF 14-11-00099 and 14-15-00879, by RFBR 12-01-00216-a, and by DAAD scholarship A/08/79920.

http://dx.doi.org/10.1016/j.acha.2014.02.002
1063-5203/

In this paper we focus on a property of good localization of both periodic wavelet functions and their Fourier coefficients. The quantitative characteristic of this property is an uncertainty constant (UC). Originally, the concept of the UC was introduced for the real line case in 1927 (see Definition 1) by Heisenberg in [12]. Its periodic counterpart was introduced in 1985 by Breitenberger in [3] (see Definition 2). The smaller UC corresponds to the better localization. In both cases there exists a universal lower bound for the UC (the uncertainty principle). In non-periodic setup the minimum is attained on the Gaussian function. But there is no periodic function attending the lower bound. So, to find a sequence of periodic functions having an asymptotically minimal UC and some additional setup, for example a wavelet structure, is a natural concern.

There is a connection between the Heisenberg and the Breitenberger UCs for wavelets. In [23] it is proved that for periodic wavelets generated by periodization (see the definition in Section 4) of a wavelet function on the real line the periodic UC tends to the real line UC of the original function as a parameter of periodization tends to infinity. It would be a possible way to construct an optimal periodic wavelet system using the periodization of a wavelet system on the real line. However, in [2] and [1] the following result is proven: if a real line function ψ generates a wavelet Bessel set and the frequency center $\omega_{0,\widehat{\psi^0}} = (\psi', \psi)_{L_2(\mathbb{R})} = 0$ (see notation $\omega_{0,\widehat{\psi^0}}$ in Definition 1), then the Heisenberg UC is greater or equal to $3/2$. Moreover, it is unknown if there exists a real line orthonormal wavelet basis or tight frame possessing the Heisenberg UC less than 2.134. This value is attained for a Daubechies wavelet [7]. The smallest possible value of the Heisenberg UC for the family of the Meyer wavelets equals to 6.874 [17]. It is well known [5] that the Heisenberg UC of the Battle–Lemarie and the Daubechies wavelets tends to infinity as their orders grow. A set of real line orthogonal wavelet bases with the uniformly bounded Heisenberg UCs as their orders (smoothness) grow is constructed in [15,16]. On the other hand, there are examples of real line wavelet frames possessing asymptotically optimal UC such as nonorthogonal B-spline wavelets [28] and their generalizations [11]. However, these frames are not tight and we are looking for an orthogonal basis or tight frame. We will discuss a particular issue of periodization in Section 4.

Some papers dealing with periodic UCs directly include [10,22,24,25]. For the first time in [25] periodic UCs uniformly bounded with respect to an MRA parameter are computed for so-called trigonometric wavelets (see also [24]). In [10], it is shown that the UCs of uniformly local, regular, and stable periodic scaling functions and wavelets are uniformly bounded. In [22] an example of an asymptotically optimal set of periodic functions $\{\varphi_h\}_{h>0}$ is constructed, namely $UC(\varphi_h) < 1/2 + \sqrt{h}/2$. Later, φ_h is used as a scaling function to generate a stationary interpolatory MRA (V_n). For the corresponding wavelet functions $\psi_{n,h}$ the UC is optimal for a fixed space V_n, but the estimate is nonuniform with respect to n, namely $UC(\psi_{n,h}) < 1/2 + 1.1n^2\sqrt{h}$. Nothing changes after orthogonalization: $UC(\psi^{\perp}_{n,h}) < 1/2 + 1.1n^2\sqrt{h}$, $UC(\varphi^{\perp}_{n,h}) < 1/2 + n^2\sqrt{h}$.

The main contribution of this paper is Theorem 4, where we construct a family of scaling sequences $\Phi^0 = \{(\varphi^a_j)_j\colon\ a > 1\}$ generating a family of wavelet sequences $\Psi^0 = \{(\psi^a_j)_j\colon\ a > 1\}$ corresponding to a nonstationary periodic MRA as it is defined in [8,14], and [27]. For a fixed level j of the MRA (V_{2^j}), similar to the construction in [22], the UCs of φ^a_j and ψ^a_j are asymptotically optimal, that is

$$\lim_{a\to\infty} \sup_{j\in\mathbb{N}} UC\big(\varphi^a_j\big) = \frac{1}{2}, \qquad \lim_{a\to\infty} UC\big(\psi^a_j\big) = \frac{1}{2}.$$

But now, for a fixed value of the parameter $a > 1$, the scaling sequence has the asymptotically optimal UC, and the wavelet sequence has the smallest currently known value of the UC for the periodic wavelet frames setup, that is

$$\lim_{j\to\infty} \sup_{a>1} UC\big(\varphi^a_j\big) = \frac{1}{2}, \qquad \lim_{j\to\infty} UC\big(\psi^a_j\big) = \frac{3}{2}.$$

As it is indicated above, the functions constructed in [22] do not have this property.