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Sequences with minimal time–frequency uncertainty ☆

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ABSTRACT

A central problem in signal processing and communications is to design signals that are compact both in time and frequency. Heisenberg's uncertainty principle states that a given function cannot be arbitrarily compact both in time and frequency, defining an "uncertainty" lower bound. Taking the variance as a measure of localization in time and frequency, Gaussian functions reach this bound for continuous-time signals. For sequences, however, this is not true; it is known that Heisenberg's bound is generally unachievable. For a chosen frequency variance, we formulate the search for "maximally compact sequences" as an exactly and efficiently solved convex optimization problem, thus providing a sharp uncertainty principle for sequences. Interestingly, the optimization formulation also reveals that maximally compact sequences are derived from Mathieu's harmonic cosine function of order zero. We further provide rational asymptotic expansions of this sharp uncertainty bound. We use the derived bounds as a benchmark to compare the compactness of well-known window functions with that of the optimal Mathieu's functions.

1. Introduction

Suppose you are asked to design filters that are sharp in the frequency domain and at the same time compact in the time domain. The same problem is posed in designing sharp probing basis functions with compact frequency characteristics. In order to formulate these problems mathematically, we need to have a correct and universal definition of compactness and clarify what we mean by saying a signal is spread in time or frequency.

These notions are well defined and established for continuous-time signals [13,34] and their properties are studied thoroughly in the literature. For such signals, we can define the time and frequency characteristics of a signal as in Table 1. Note the connection of these definitions with the mean and variance of a probability distribution function $|x(t)|^2/\|x\|^2$. The value of Δ_t^2 is considered as the spread of the signal in the time

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Table 1
Time and frequency centers and spreads for a continuous time signal $x(t)$.

Domain	Center	Spread
Time	$\mu_t = \frac{1}{\|x\|^2}\int_{t\in\mathbb{R}} t\|x(t)\|^2 dt$	$\Delta_t^2 = \frac{1}{\|x\|^2}\int_{t\in\mathbb{R}}(t-\mu_t)^2\|x(t)\|^2 dt$
Frequency	$\mu_{\omega_c} = \frac{1}{2\pi\|x\|^2}\int_{\omega\in\mathbb{R}} \omega\|X(\omega)\|^2 d\omega$	$\Delta_{\omega_c}^2 = \frac{1}{2\pi\|x\|^2}\int_{\omega\in\mathbb{R}}(\omega-\mu_\omega)^2\|X(\omega)\|^2 d\omega$

Table 2
Time and frequency centers and spreads for a discrete time signal x_n as extensions of Table 1 [34].

Domain	Center	Spread
Time	$\mu_n = \frac{1}{\|x\|^2}\sum_{k\in\mathbb{Z}} k\|x_k\|^2$	$\Delta_n^2 = \frac{1}{\|x\|^2}\sum_{k\in\mathbb{Z}}(k-\mu_n)^2\|x_k\|^2$
Frequency	$\mu_{\omega_\ell} = \frac{1}{2\pi\|x\|^2}\int_{-\pi}^{\pi} \omega\|X(e^{j\omega})\|^2 d\omega$	$\Delta_{\omega_\ell}^2 = \frac{1}{2\pi\|x\|^2}\int_{-\pi}^{\pi}(\omega-\mu_\omega)^2\|X(e^{j\omega})\|^2 d\omega$

domain while $\Delta_{\omega_c}^2$ represents its spread in the frequency domain. We say that a signal is compact in time (or frequency) if it has a small time (or frequency) spread.

The Heisenberg uncertainty principle [13,27,28] states that continuous-time signals cannot be arbitrarily compact in both domains. Specifically, for any $x(t) \in L^2(\mathbb{R})$,

$$\eta_c = \Delta_t^2 \Delta_{\omega_c}^2 \geq \frac{1}{4}, \tag{1}$$

where the lower bound is achieved for Gaussian signals of the form $x(t) = \gamma e^{-\alpha(t-t_0)^2 + j\omega_0 t}$, $\alpha > 0$ [9]. The subscript c stands for continuous-time definitions. We call η_c the *time–frequency spread* of x.

Although the continuous Heisenberg uncertainty principle is widely used in theory, in practice we often work with discrete-time signals (e.g. filters and wavelets). Thus, equivalent definitions for discrete-time sequences are needed in signal processing. In the next section we study two common definitions of center and spread available in the literature.

1.1. Uncertainty principles for sequences

An obvious and intuitive extension of the definitions in Table 1 for discrete-time signals is presented in Table 2, where

$$X\left(e^{j\omega}\right) = \sum_{k\in\mathbb{Z}} x_k e^{-j\omega k} \quad \omega \in \mathbb{R}, \tag{2}$$

is the discrete-time Fourier transform (DTFT) of x_n.

Using the definitions in Table 2 [34], we can also state the Heisenberg uncertainty principle for discrete-time signals as

$$\eta_\ell = \Delta_n^2 \Delta_{\omega_\ell}^2 > \frac{1}{4}, \quad x_n \in \ell^2(\mathbb{Z}) \text{ with } X\left(e^{j\pi}\right) = 0, \tag{3}$$

where the subscript ℓ stands for *linear* in reference to the definition of the frequency spread. Note the extra assumption on the Fourier transform of the signal in (3). This assumption is necessary for the result to hold.

Example 1. Take $x_n = \delta_n + 7\delta_{n-1} + 2\delta_{n-2}$. It is easy to verify that $|X(e^{j\pi})| = 0.22 \neq 0$, which violates the condition $X(e^{j\pi}) = 0$. The linear time–frequency spread of this signal according to Table 2 is $\eta_\ell = 0.159 < 1/4$.