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Letter to the Editor

Convergence analysis for iterative data-driven tight frame construction scheme

Chenglong Bao, Hui Ji *, Zuowei Shen

Department of Mathematics, National University of Singapore, Singapore 117543, Singapore

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ABSTRACT

Sparse modeling/approximation of images plays an important role in image restoration. Instead of using a fixed system to sparsely model any input image, a more promising approach is using a system that is adaptive to the input image. A non-convex variational model is proposed in [1] for constructing a tight frame that is optimized for the input image, and an alternating scheme is used to solve the resulting non-convex optimization problem. Although it showed good empirical performance in image denoising, the proposed alternating iteration lacks convergence analysis. This paper aims at providing the convergence analysis of the method proposed in [1]. We first established the sub-sequence convergence property of the iteration scheme proposed in [1], i.e., there exists at least one convergent sub-sequence and any convergent sub-sequence converges to a stationary point of the minimization problem. Moreover, we showed that the original method can be modified to have sequence convergence, i.e., the modified algorithm generates a sequence that converges to a stationary point of the minimization problem.

1. Introduction

It is now well established that sparse modeling is a very powerful tool for many image recovery tasks, which models an image as the linear combination of only a small number of elements of some system. Such a system can be either a basis or an over-complete system. When using the sparsity prior of images to regularize image recovery, the performance largely depends on how effective images of interest can be sparsely approximated under the given system. Therefore, a fundamental question in sparsity-based image regularization is how to define a system such that the target image has an optimal sparse approximation. Earlier work on sparse modeling focuses on the design of orthonormal bases, such as *discrete cosine transform* [2], *wavelets* [3,4]. Owing to their better performance in practice, over-complete systems have been more recognized in sparsity-based image recovery methods. In particular, as a redundant extension of orthonormal

* Corresponding author.
E-mail addresses: baochenglong@nus.edu.sg (C. Bao), matjh@nus.edu.sg (H. Ji), matzuows@nus.edu.sg (Z. Shen).

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bases, tight frames are now wide-spread in many applications as they have the same efficient and simple decomposition and reconstruction schemes as orthonormal bases. Many types of tight frames have been proposed for sparse image modeling including *shift-invariant wavelets* [5], *framelets* [6,7], *curvelets* [8] and many others. These tight frames are optimized for the signals with certain functional properties, which do not always hold true for natural images. As a consequence, a more effective approach to sparsely approximate images of interest is to construct tight frames that are adaptive to the inputs.

In recent years, the concept of data-driven systems has been exploited to construct adaptive systems for sparsity-based modeling (see e.g. [1,9–11]). The basic idea is to construct the system that is adaptive to the input so as to obtain a better sparse approximation than the predefined ones. Most sparsity-based dictionary learning methods [9–11] treat the input image as the collection of small image patches, and then construct an over-complete dictionary for sparsely approximating these image patches. Despite the impressive performance in various image restoration tasks, the minimization problems proposed by these methods are very challenging to solve. As a result, the numerical methods proposed in past for these models not only lack rigorous analysis on their convergence and stability, but also are very computationally demanding.

Recently, Cai et al. [1] proposed a variational model to learn a tight frame system that is adaptive to the input image in terms of sparse approximation. Differently from the existing over-complete dictionary learning methods, the adaptive systems constructed in [1] are tight frames that have *perfect reconstruction property*, a property ensuring that any input can be perfectly reconstructed by its canonical coefficients in a simple manner. The tight frame property of the system constructed in [1] not only is attractive to many image processing tasks, but also leads to very efficient construction scheme. Indeed, by considering a special class of tight frames, the construction scheme proposed in [1] only requires solving an ℓ_0 norm related non-convex minimization problem:

$$\min_{D\in\mathbb{R}^{m\times m},C\in\mathbb{R}^{m\times n}} \left\|C - D^\top Y\right\|_F^2 + \lambda_0^2\|C\|_0, \quad \text{s.t.} \quad D^\top D = m^{-1} I_m, \tag{1}$$

where D contains framelet filters and C contains the canonical frame coefficients. An alternating iteration is proposed in [1] for solving (1), which is very fast as both sub-problems in each iteration have closed-form solutions. It is shown that, with comparable performance in image denoising, the proposed adaptive tight frame construction runs much faster than other generic dictionary learning methods (e.g. the K-SVD method [10]). However, Cai et al. [1] did not provide any convergence analysis of the proposed method.

As a sequel to [1], this paper provides the convergence analysis of the alternating iterative method proposed in [1] for solving (1). In this paper, we showed that the algorithm provided by [1] has sub-sequence convergence property. In other words, we showed that there exists at least one convergent sub-sequence of the sequence generated by the algorithm [1] and any convergent sub-sequence converges to a stationary point of (1). Moreover, we empirically observed that the sequence generated by the algorithm proposed in [1] itself is not convergent. Motivated by the theoretical interest, we modified the algorithm proposed in [1] by adding a proximal term in the iteration scheme, and then showed that the modified algorithm has sequence convergence. In other words, the sequence generated by the modified method convergences to a stationary point of (1).

2. Brief review on data-driven tight frame construction and related works

In this section, we gave a brief review on tight frames, data-driven tight frames proposed in [1] and some most related works. Interested readers are referred to [12,13] for more details.