Appl. Comput. Harmon. Anal. 36 (2014) 198–214

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

From dual pairs of Gabor frames to dual pairs of wavelet frames and vice versa

Ole Christensen [a], Say Song Goh [b,*]

[a] Technical University of Denmark, Department of Applied Mathematics and Computer Science, Building 303, 2800 Lyngby, Denmark
[b] Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Republic of Singapore

ARTICLE INFO

Article history:
Received 18 June 2012
Received in revised form 24 January 2013
Accepted 29 March 2013
Available online 8 April 2013
Communicated by Pierre G. Lemarié-Rieusset

Keywords:
Gabor frames
Wavelet frames
Dual frames
Meyer's wavelet
Shannon's wavelet

ABSTRACT

We discuss an elementary procedure that allows us to construct dual pairs of wavelet frames based on certain dual pairs of Gabor frames and vice versa. The construction preserves tightness of the involved frames. Starting with Gabor frames generated by characteristic functions the construction leads to a class of tight wavelet frames that include the Shannon (orthonormal) wavelet, and applying the construction to Gabor frames generated by certain exponential B-splines yields wavelet frames generated by functions whose Fourier transforms are compactly supported splines with geometrically distributed knot sequences. On the other hand, the pendant of the Meyer wavelet turns out to be a tight Gabor frame generated by a $C^\infty(\mathbb{R})$ function with compact support. As an application of our results we show that for each given pair of bandlimited dual wavelet frames it is possible to construct dual wavelet frames for any desired scaling and translation parameters.

1. Introduction

At the beginning of the "wavelet era", wavelet analysis and Gabor analysis were treated in a parallel way, where results and methods in one of the settings would often have an immediate impact on the other. This is illustrated by the work by Daubechies, Grossmann and Meyer [9], Daubechies [7,8], and Heil and Walnut [16] that contain sections on wavelet analysis as well as Gabor analysis. Also, the Feichtinger–Gröchenig theory for atomic decomposition [10] lays a common foundation for wavelet expansions and Gabor analysis.

From 1990, wavelet analysis became focused on multiscale construction that has no pendant in Gabor analysis. Also the trends in Gabor analysis went new ways, e.g., into studies of special classes of operators (spreading operators, pseudodifferential operators, among others; see [12,13] and the references therein), as well as studies of frame properties for Gabor windows versus dual windows [15]. Thus, practically since 1990, wavelet analysis and Gabor analysis have been two separate research areas, with little impact on each other.

The purpose of this paper is to reestablish the connection between the two topics, taking the subsequent developments into account. We will provide a procedure that allows us to construct dual pairs of wavelet frames based on certain dual pairs of Gabor frames, and vice versa. We begin, in Section 2, with the introduction of a transform that will move the Gabor structure (to be defined below) into the wavelet structure. Based on this we show in Section 2.1 how to construct bandlimited wavelet frames (i.e., wavelet frames generated by functions with compactly supported Fourier transform) based on Gabor frames with compactly supported window functions. For example, an application to Gabor frames generated by the characteristic functions $\chi_{[N-1,N]}$, $N \in \mathbb{R}$, leads to a class of tight wavelet frames that includes Shannon's wavelet. Other

* Corresponding author.
E-mail addresses: ochr@dtu.dk (O. Christensen), matgohss@nus.edu.sg (S.S. Goh).

1063-5203/$ – see front matter
http://dx.doi.org/10.1016/j.acha.2013.03.005

explicit constructions based on Gabor frames generated by exponential B-splines are provided in Section 2.2; the outcome is a class of attractive dual wavelet frame pairs generated by functions whose Fourier transforms are compactly supported splines with geometrically distributed knots. In Section 2.3 we introduce extra degrees of freedom that allow us to change the parameters of the constructed frames.

In Section 3 we consider the analogous problem of constructing Gabor frames based on wavelet frames. We first introduce the transformation that moves the wavelet structure into the Gabor structure. In its basic form, this transform appears in the fundamental paper [9] by Daubechies, Grossmann and Meyer (see Section 3 for a description). In Section 3.1 we present the results about how to construct Gabor frames with compactly supported windows based on bandlimited wavelet frames. The special case of tight frames is considered in Section 3.2. In Section 3.3 we introduce extra degrees of freedom that allow us to change the parameters of the constructed frames. As a result, we can then construct a Gabor frame based on the Meyer wavelet; it turns out to be a tight frame generated by a $C^\infty(\mathbb{R})$ function with compact support. In Section 3.4 we highlight an instance of gaining additional insight to bandlimited wavelet frames by applying our transforms between wavelet frames and Gabor frames. It shows that for such dual wavelet frames, there is an elementary way to construct other wavelet frames with desired (and arbitrary) dilation and translation parameters.

In the rest of this introduction we review some results from frame theory. Let $\mathcal{H}$ denote a separable Hilbert space. A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{i\in I}\left|\langle f, f_i\rangle\right|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1.1}$$

The constants A and B are *frame bounds*. The sequence $\{f_i\}_{i\in I}$ is a *Bessel sequence* if at least the upper bound in (1.1) is satisfied. A frame is *tight* if we can choose $A = B$ in (1.1). For any frame $\{f_i\}_{i\in I}$ there exists at least one *dual frame*, i.e., a frame $\{\widetilde{f}_i\}_{i\in I}$ for which

$$f = \sum_{i\in I}\langle f, f_i\rangle\widetilde{f}_i, \quad \forall f \in \mathcal{H}. \tag{1.2}$$

We will consider Gabor frames and wavelet frames, both in the Hilbert space $L^2(\mathbb{R})$. A *Gabor system* in $L^2(\mathbb{R})$ has the form $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ for some parameters $a, b > 0$ and a given function $g \in L^2(\mathbb{R})$. Using the *translation operators* $T_a f(x) := f(x-a)$, $a \in \mathbb{R}$, and the *modulation operators* $E_b f(x) := e^{2\pi ibx}f(x)$, $b \in \mathbb{R}$, both acting on $L^2(\mathbb{R})$, we will denote a Gabor system by $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.

A *wavelet system* in $L^2(\mathbb{R})$ has the form $\{a^{j/2}\psi(a^jx - kb)\}_{j,k\in\mathbb{Z}}$ for some parameters $a > 1$, $b > 0$ and a given function $\psi \in L^2(\mathbb{R})$. Introducing the *scaling operators* $(D_a f)(x) := a^{1/2}f(ax)$, $a > 0$, acting on $L^2(\mathbb{R})$, the wavelet system can be written as $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$.

The key tools in our approach are equations characterizing dual frame pairs in the Gabor setting and the wavelet setting. For the readers' convenience we state both of them here.

The duality conditions for two Gabor systems were found by Ron and Shen [23,24]. We use the formulation due to Janssen [17]:

Theorem 1.1. *Given $b, \alpha > 0$, two Bessel sequences $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\widetilde{g}\}_{m,n\in\mathbb{Z}}$, where $g, \widetilde{g} \in L^2(\mathbb{R})$, form dual Gabor frames for $L^2(\mathbb{R})$ if and only if for all $n \in \mathbb{Z}$,*

$$\sum_{j\in\mathbb{Z}}\overline{g(x+j\alpha)}\widetilde{g}(x+j\alpha+n/b) = b\delta_{n,0}, \quad \text{a.e. } x \in \mathbb{R}. \tag{1.3}$$

Note that for $n = 0$, the condition (1.3) amounts to

$$\sum_{j\in\mathbb{Z}}\overline{g(x+j\alpha)}\widetilde{g}(x+j\alpha) = b, \quad \text{a.e. } x \in \mathbb{R}. \tag{1.4}$$

If the functions g and $\widetilde{g}$ are compactly supported, the conditions in (1.3) are automatically satisfied for $n \neq 0$ whenever $b > 0$ is sufficiently small. Thus, in that case it is enough to check (1.4). For more general information on Gabor analysis, we refer to the monograph [14] by Gröchenig and the compiled volumes [12,13].

We now state the characterizing equations for dual wavelet frames; see [5]. Here $\widehat{f}$ denotes the Fourier transform for $f \in L^1(\mathbb{R})$ defined by $\widehat{f}(\gamma) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i\gamma x}\,dx$, and extended to $L^2(\mathbb{R})$ in the usual way.

Theorem 1.2. *Given $a > 1$, $b > 0$, two Bessel sequences $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$, where $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$, form dual wavelet frames for $L^2(\mathbb{R})$ if and only if the following two conditions are satisfied:*

(i) $\sum_{j\in\mathbb{Z}}\overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma) = b$ *for a.e.* $\gamma \in \mathbb{R}$.