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Identification of stochastic operators

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ABSTRACT

Based on the here developed functional analytic machinery we extend the theory of operator sampling and identification to apply to operators with stochastic spreading functions. We prove that identification with a delta train signal is possible for a large class of stochastic operators that have the property that the autocorrelation of the spreading function is supported on a set of 4D volume less than one and this support set does not have a defective structure. In fact, unlike in the case of deterministic operator identification, the geometry of the support set has a significant impact on the identifiability of the considered operator class. Also, we prove that, analogous to the deterministic case, the restriction of the 4D volume of a support set to be less or equal to one is necessary for identifiability of a stochastic operator class.

1. Introduction

In the fields of wireless communication and radar and sonar acquisition, a sounding signal that is known both to the sender and to the observer is sent into the channel in order to determine the characteristics of the channel from the received echo. Similarly, in control theory, the problem of identifying a system from the output to a given input is called system identification.

Both deterministic and stochastic operator identification have their roots in the works of Kailath [1] and Bello [2]. They suggested a criterion for identifiability based on the *spread* of the operator, defined as the area of the support of the spreading function. In [3,4] the criterion that it is necessary and sufficient that the spread must be less than one for a deterministic operator to be identifiable has been theoretically verified and justified, giving new life to this engineering dogma. The universal boundary of *one* for the spread of the deterministic operator to be identifiable is closely related to the Heisenberg uncertainty principle of quantum mechanics. This connection is evident in the time–frequency analysis nature of the proofs given in [3,4], as these rely on the representation theory of the Weyl–Heisenberg group.

The utility of weighted delta trains as a theoretical tool for the study of identification and sampling theory stems from their position as infinite bandwidth unbounded temporal support sounding signals. Recently, more practical identifier signals for a class of channels with a parametric model on the channel structure have been discovered [5,6]. In another development, it was shown that a stiff requirement to know the support of the spreading function (precisely the set whose area must be less than one) prior to sounding can be removed by using compressed sensing techniques [7,8].

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Here, we continue to rely on weighted delta trains as identifiers to develop a parallel theory of identification of operators with stochastic spreading functions. Since stochastic operators include deterministic operators as special case, it is tenable to suppose that in some form the restriction on the spread to be less than one retains its relevance. However the rules of the game change, as the object to recover, the spreading function, belongs to a much larger class of objects. The common strategy to circumvent these difficulties is to decrease the complexity of the channel by requiring the spreading function to have a degenerate form, stationarity in both the time and frequency variables, which corresponds to a WSSUS channel. The study is hereby reduced to the recovery of the so-called scattering function, a deterministic function in two variables, defined below in (2). Even in this simplified setting, the communication engineering literature still seems to accept the insight of Bello that the area of the support of the scattering function is a necessary requirement for the identifiability of the operator [2]. In fact, even this characterization of the fading properties of a channel is discarded in favor of a simpler yet *spread factor* given by the area of the minimum rectangle that encompasses the support of the spreading function (and hence, scattering function) in the time–frequency plane. This rule of thumb is perpetuated in the classical books as recent as the monograph of Proakis [9].

In [10], we argue that as in the deterministic case, the condition for the spread factor to be less than one is sufficient in the case of identifiable WSSUS channels, and establish the direct applicability of time–frequency analysis techniques of Kozek, Pfander and Walnut in this simplified stochastic setting. In [11], we assume functional analytic results proven here and give a detailed analysis of the general case of stochastic operator sampling with a fully stochastic spreading function. The work in [11] extends sampling results for operators and discusses their applications. A surprising corollary shows that using weighted delta trains as identifiers for WSSUS channels allows the recovery of the scattering function from the autocorrelation of the received signal irrespective of the area of the support of the scattering function.

Here, we settle the question of the identifiability of stochastic operators with a general necessary and sufficient Theorem 4.3. It turns out that the volume of the set M, the support of the stochastic spreading function, alone is not enough for identifiability of the corresponding operator. The geometry of the set M plays an important role for the possibility of identification.

The popular and herein used model for channels and linear time-variant (LTV) systems is

$$(Hf)(x) = \iint \eta(t,\nu)\, M_\nu T_t f(x)\, \mathrm{d}t\, \mathrm{d}\nu, \tag{1}$$

where T_t is a *time-shift* by t, that is, $T_t f(x) = f(x-t)$, $t \in \mathbb{R}$, and M_ν is a *frequency shift* or *modulation* given by $M_\nu f(x) = e^{2\pi i\nu x} f(x)$, $\nu \in \mathbb{R}$. Taking Fourier transforms, it follows that $\widehat{M_\nu f}(\xi) = \widehat{f}(\xi-\nu) = T_\nu \widehat{f}(\xi)$ for all $\xi \in \mathbb{R}$. The function $\eta(t,\nu)$ is called the *(Doppler-delay) spreading function* of H. Classically, the domain and codomain of H are taken to be the Lebesgue space of square integrable functions $L^2(\mathbb{R})$ or the discrete finite-dimensional space $\mathbb{C}_L$. More generally, $f(x)$, $\eta(t,\nu)$ and $g(x) = Hf(x)$ can be elements in spaces of generalized functions, such as the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, the continuous dual of the space $\mathcal{S}(\mathbb{R}^d)$ of infinitely differentiable rapidly decaying functions.

It is common that models of wireless channels and radar environments take the stochastic nature of the medium into account. In such models, the spreading function $\boldsymbol{\eta}(t,\nu)$ or the sounding signal $\boldsymbol{f}(x)$ in (1), or both, are random processes (that will henceforth be denoted by boldface letters) such that, for example, every sample of the spreading function $\boldsymbol{\eta}(t,\nu;\omega)$ belongs to one of the spaces mentioned above. In this paper we consider only the spreading function to be stochastic, leaving the sounding signal completely deterministic.

Usually, the operator is split into the sum of its deterministic portion, representing the mean behavior of the channel, and its zero-mean stochastic portion that represents the noise and the environment. We assume that this decomposition has already taken place and focus on operators with purely stochastic zero-mean spreading functions. For a treatment of the deterministic part, we refer to [3,4,12].

The statistic that presents the most interest in this setting is the *autocorrelation* of the spreading function

$$R_{\boldsymbol{\eta}}(t,\nu;t',\nu') = \mathbb{E}\big\{\boldsymbol{\eta}(t,\nu)\,\overline{\boldsymbol{\eta}(t',\nu')}\big\},$$

and we will pursue the goal of determining $R_{\boldsymbol{\eta}}$ from $R_{\boldsymbol{H}f}$, that is, $R_{\boldsymbol{\eta}}$ from the autocorrelation of the stochastic channel output $\boldsymbol{H}f$ (see Definition 3.1). The time-varying case most studied in the literature assumes the special form

$$R_{\boldsymbol{\eta}}(t,\nu;t',\nu') = C_{\boldsymbol{\eta}}(t,\nu)\,\delta(t-t')\,\delta(\nu-\nu'). \tag{2}$$

Such operators are referred to as *wide-sense stationary operators with uncorrelated scattering*, or WSSUS. The function $C_{\boldsymbol{\eta}}(t,\nu)$ is then called *scattering function* [13,14,10]. Our results do not presuppose the stationarity of $\boldsymbol{H}$, instead, they include it as an interesting special case. (See Fig. 1.)

Under the a priori assumption that the operator $H: X \to Y$ belongs to class of linear operators $\mathcal{H}$, identification of the operator from the received echo Hf is possible only if for some sounding signal f the mapping $\Phi_f : H \mapsto Hf$ is injective,

$$H_1 f = H_2 f \quad \Rightarrow \quad H_1 = H_2.$$

For linear mappings between Banach spaces to *identify* $\mathcal{H}$ in a *stable* way, we require Φ_f and its inverse to be bounded:

$$A\|H_1 - H_2\|_{\mathcal{H}} \leqslant \|H_1 f - H_2 f\|_Y \leqslant B\|H_1 - H_2\|_{\mathcal{H}} \quad \text{for all } H_1, H_2 \in \mathcal{H}. \tag{3}$$