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Letter to the Editor

Frequency domain Walsh functions and sequences: An introduction

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ABSTRACT

In this letter, a new set of orthogonal band-limited basis functions is introduced. This set of basis functions is derived from the inverse Fourier transform of the frequency domain Walsh functions. The Fourier transforms of the Walsh functions were calculated by Siemens and Kitai in 1973 but they have been overlooked in the literature. Some of the properties of these functions are studied in this paper. Moreover, the orthogonal discrete version of these functions is obtained by truncation, sampling and orthogonalization utilizing the orthogonal Procrustes problem.

1. Introduction

The most well-known band-limited orthogonal function set is the basis functions for the shift-invariant Hilbert space of the sampling process. This Hilbert space is defined as $\mathcal{H} = \text{span}\{\phi_n(t) = \phi(t-n)\}_{n\in\mathbb{Z}}$. Shannon, in his seminal paper [1], showed that the generating function of this orthogonal basis function set ($\phi_0(t)$) is the sinc function. The orthogonal projection of a band-limited function $f(t)$ on this Hilbert space (c_n) is, in fact, a sampled version of the function or $f(n)$. Several other sampling paradigms based on shift-invariant and wavelet spaces are also introduced [2].

Orthogonal decomposition of band-limited functions can also be performed by generalization of the concept behind the sinc function. The Hilbert transform of the sinc function can be easily derived as $(1-\cos(\pi t))/(\pi t)$ and it is called the cosc function [3] or cosinc function [4]. The cosc function is obviously an odd function but its magnitude Fourier transform is the same as that of the sinc function. A combination of these two function creates a basis function that spans the band-limited functions in L_2 (Paley–Wiener space). This band-limited orthogonal basis function set is used in speech coding [5] and Ultra-wide Band pulse generation [6].

In this letter a new orthogonal band-limited function set is introduced, which is derived from the inverse Fourier transform of Walsh functions. We call these functions frequency domain Walsh functions (FDWF). The Walsh orthogonal function set was introduced by J.L. Walsh in 1923 [7] and it has been extensively used in different applications. The Fourier transforms of these functions were found in 1973 by Siemens and Kitai [8] but they have been completely overlooked in both mathematics and engineering until now.

To find the discrete version of the FDWFs, we can sample and truncate them. Simple orthogonalization can produce discrete version of the orthogonal FDWFs. To make them as close as possible to FDWFs, we can use the orthogonal Procrustes problem. The resultant functions are called frequency domain Walsh sequences (FDWS).

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2. Frequency-domain Walsh functions (FDWF)

If we consider the Walsh functions in the frequency domain, we will have a set of orthogonal functions whose Fourier transforms are Walsh functions. To the extent of our knowledge, Walsh functions have always been used in the time domain (e.g., [9,10]) and no one has ever used them in the frequency domain.

Walsh functions are derived by a recursive formula or by using Haar functions [9]. In [11], a non-recursive formula for deriving Walsh functions is introduced. Based on this non-recursive formula, the Fourier transform of Walsh functions is found in [8]. We can use the duality of the Fourier transform to derive the inverse Fourier transform of the Walsh functions when they are defined in the frequency domain. The following Fourier transform pair is the result of this derivation:

$$\begin{aligned}
&\phi_m(t) \quad \Leftrightarrow \quad \Phi_m(\omega),\\
&\phi_m(t) = \frac{\omega_c(-1)^{g_0}}{\pi}\left[\prod_{k=0}^{M-1}\cos\left(\frac{\omega_c t}{2^{k+1}} - \frac{\pi g_k}{2}\right)\right]\operatorname{sinc}\left(\frac{\omega_c t}{\pi 2^M}\right),\\
&\Phi_m(\omega) = (j)^{\alpha}(-1)^m W_m(\omega),
\end{aligned} \tag{1}$$

where $\phi_m(t)$ are the FDWFs, $W_m(\omega)$ is the Walsh function of order m, M is the number of bits representing m, $G = \overline{g_{M-1}g_{M-2}\dots g_1g_0}$ is the Gray code representation of m, g_k is the kth bit of G, ω_c is the bandwidth of the functions and α is the number of Gray code bits of value ONE in G. In Fig. 1, the even FDWFs of order 0, 4, 8, 12 and the odd FDWFs of order 1, 5, 9, and 13 are depicted. Since Walsh functions construct a complete orthogonal function set [7], the FDWFs also construct a complete orthogonal function set. The dimension of this set is infinite; thus, we can obtain an infinite number of orthogonal bandlimited functions. Some of the properties of FDWFs are studied in the next section.

3. Properties of the FDWFs

3.1. Symmetry

It is obvious that $\phi_{2k}(t)$ and $\phi_{2k+1}(t)$ for $k = 0, 1, 2, \dots$ are, respectively, even and odd functions. It is worth noting that $\phi_0(t)$ and $\phi_1(t)$ are respectively the sinc and cosc functions or $\phi_0(t) = \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t/\pi)$ and $\phi_1(t) = \frac{\omega_c}{\pi}\operatorname{cosc}(\omega_c t/\pi)$.

3.2. Behavior at the origin

Theorem 1. *If $\phi_m(t)$ is the FDWF of order m, then $\phi_m(0) = \delta_{mm}$, where δ_{mm} is the Kronecker delta.*

Proof. If $m \neq 0$, at least one of the bits of the Gray code representation of m will not be zero. This means at least one of the cosine terms in the product of cosines in (1) when $t = 0$ will be $\cos(-\pi/2) = 0$, which is sufficient to annihilate the equation. On the other hand, when $m = 0$, $\phi_0(t)$ is a sinc function; thus, $\phi_0(0) = 1$. □

Theorem 2. *The first derivative of the FDWF of order m at the origin is zero if the order of the function is not $2^M - 1$, where M is the number of bits representing the order m.*

Proof. By considering the fact that the first derivative of the sinc function at $t = 0$ is zero, we can find the first derivative of $\phi_m(t)$ at $t = 0$ as

$$\left.\frac{d\phi_m(t)}{dt}\right|_{t=0} = \frac{\omega_c^2(-1)^{g_0}}{\pi}\sum_{\ell=0}^{M-1}\left[\frac{1}{2^{\ell+1}}\sin\left(\frac{\pi g_\ell}{2}\right)\prod_{\substack{k=0\\k\neq\ell}}^{M-1}\cos\left(\frac{\pi g_k}{2}\right)\right]. \tag{2}$$

It can be seen in (2) that if there is only one g_k bit whose value is ONE, one of the term of the summation will remain nonzero, thus the first derivative of $\phi_m(t)$ is not zero at $t = 0$. Otherwise, if there is more than one g_k bit whose values are ONE, there will be at least one cosine in the product term of (2) whose angle is $\frac{\pi}{2}$ in each terms of the summation. This means that every term of the summation is certainly annihilated, thus the first derivative of $\phi_m(t)$ is zero at $t = 0$. According to the Gray code structure, the numbers whose Gray codes have only one bit of value ONE are $m = 1, 3, 7, 15, \dots$ or $m = 2^M - 1$, where M is the number of bits representing the order m. □

3.3. Zero crossings

Theorem 3. *The zeros of the FDWSs ($\phi_m(t)$) occur at $t = \frac{\pi\ell 2^M}{\omega_c}$ and $t = \frac{(2\ell+1+g_k)\pi 2^k}{\omega_c}$ for $\ell \in \mathbb{Z}$ and $k = 0, 1, 2, \dots, M-1$.*