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Stable restoration and separation of approximately sparse signals

Christoph Studer *, Richard G. Baraniuk **

Department of Electrical and Computer Engineering, Rice University, 6100 Main Street, Houston, TX 77005, USA

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ABSTRACT

This paper develops new theory and algorithms to recover signals that are approximately sparse in some general dictionary (i.e., a basis, frame, or over-/incomplete matrix) but corrupted by a combination of interference having a sparse representation in a second general dictionary and measurement noise. The algorithms and analytical recovery conditions consider varying degrees of signal and interference support-set knowledge. Particular applications covered by the proposed framework include the restoration of signals impaired by impulse noise, narrowband interference, or saturation/clipping, as well as image in-painting, super-resolution, and signal separation. Two application examples for audio and image restoration demonstrate the efficacy of the approach.

1. Introduction

We investigate the *recovery* problem of the coefficient vector $\mathbf{x} \in \mathbb{C}^{N_a}$ from the corrupted M-dimensional observations

$$\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{e} + \mathbf{n}, \quad (1)$$

where $\mathbf{A} \in \mathbb{C}^{M \times N_a}$ and $\mathbf{B} \in \mathbb{C}^{M \times N_b}$ are general deterministic dictionaries; examples for general dictionaries include bases, frames, or over-/incomplete matrices whose columns have unit Euclidean (or ℓ_2) norm. The vector $\mathbf{x}$ is assumed to be *approximately sparse*, i.e., its main energy (in terms of the sum of absolute values, for example) is concentrated in only a few entries. The M-dimensional signal vector is defined as $\mathbf{y} = \mathbf{A}\mathbf{x}$. The vector $\mathbf{e} \in \mathbb{C}^{N_a}$ represents interference and is assumed to be *perfectly sparse*, i.e., only a few entries are nonzero, and $\mathbf{n} \in \mathbb{C}^M$ corresponds to measurement noise. Apart from the bound $\|\mathbf{n}\|_2 < \varepsilon$, the measurement noise is arbitrary. We emphasize that the interference and noise components $\mathbf{e}$ and $\mathbf{n}$ can depend on the vector $\mathbf{x}$ and/or the dictionary $\mathbf{A}$.

The setting (1) also allows us to study *signal separation*, i.e., the separation of two distinct features $\mathbf{A}\mathbf{x}$ and $\mathbf{B}\mathbf{e}$ from the noisy observation $\mathbf{z}$. Here, the vector $\mathbf{e}$ in (1) is also allowed to be approximately sparse and is used to represent a second desirable feature (rather than undesired interference). Signal separation amounts to simultaneously recovering the vectors $\mathbf{x}$ and $\mathbf{e}$ from the noisy measurement $\mathbf{z}$ followed by computation of the individual signal features $\mathbf{A}\mathbf{x}$ and $\mathbf{B}\mathbf{e}$.

* Corresponding author. Fax: +1 713 348 5685.
** Principal corresponding author. Fax: +1 713 348 5685.
E-mail addresses: studer@rice.edu (C. Studer), richb@rice.edu (R.G. Baraniuk).
URLs: http://www.ece.rice.edu/~cs32/ (C. Studer), http://web.ece.rice.edu/richb/ (R.G. Baraniuk).

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1.1. Applications for the model (1)

Both the recovery and separation problems outlined above feature prominently in numerous applications (see [1–18] and the references therein), including:

- *Impulse noise:* The recovery of approximately sparse signals corrupted by impulse noise [13] corresponds to recovery of $\mathbf{x}$ from (1) by setting $\mathbf{B} = \mathbf{I}_M$ and associating the interference $\mathbf{e}$ with the impulse-noise vector. Practical examples include restoration of audio signals impaired by click/pop noise [1,2] and reading from unreliable memories [14].
- *Narrowband interference:* Audio, video, and communication signals are often corrupted by narrowband interference. A particular example is electric hum, which typically occurs in improperly designed audio or video equipment. Such impairments naturally exhibit a sparse representation in the frequency domain, which amounts to setting $\mathbf{B}$ to the inverse discrete Fourier transform matrix.
- *Saturation and clipping:* Nonlinearities in amplifiers may result in signal saturation, cf. [7,16,17]. Here, instead of the signal vector $\mathbf{y}$ of interest, one observes a saturated (or clipped) version $\mathbf{z} = \mathbf{y} + \mathbf{e} + \mathbf{n}$, where the nonzero entries of $\mathbf{e}$ correspond to the difference between the saturated signal and the original signal $\mathbf{y}$. The noise vector $\mathbf{n}$ can be used to model residual errors that are not captured by the interference component $\mathbf{Be}$.
- *Super-resolution and in-painting:* In super-resolution [3,15] and in-painting [6,8–10] applications, only a subset of the entries of the (full-resolution) signal vector $\mathbf{y} = \mathbf{Ax}$ is available. With (1), the interference vector $\mathbf{e}$ accounts for the missing parts of the signal, i.e., the locations of the nonzero entries of $\mathbf{e}$ correspond to the missing entries in $\mathbf{y}$ and are set to some arbitrary value. The missing parts of $\mathbf{y}$ are then filled in by recovering $\mathbf{x}$ from $\mathbf{z} = \mathbf{Ax} + \mathbf{e} + \mathbf{n}$ followed by computation of the (full-resolution) signal vector $\mathbf{y} = \mathbf{Ax}$.
- *Signal separation:* The framework (1) can be used to model the decomposition of signals into two distinct features. Prominent application examples are the separation of texture from cartoon parts in images [4,6,18] and the separation of neuronal calcium transients from smooth signals caused by astrocytes in calcium imaging [5]. In both applications, $\mathbf{A}$ and $\mathbf{B}$ are chosen such that each feature can be represented by approximately sparse vectors in one dictionary. Signal separation then amounts to simultaneously extracting $\mathbf{x}$ and $\mathbf{e}$ from $\mathbf{z}$, where $\mathbf{Ax}$ and $\mathbf{Be}$ represent the individual features.

In many applications outlined above, a predetermined (and possibly optimized) dictionary pair $\mathbf{A}$ and $\mathbf{B}$ is used. It is therefore of significant practical interest to identify the fundamental limits on the performance of restoration or separation from the model (1) for the deterministic setting, i.e., assuming no randomness in the dictionaries, the signal, interference, or the noise vector. Deterministic recovery guarantees for the special case of *perfectly* sparse vectors $\mathbf{x}$ and $\mathbf{e}$ and *no* measurement noise have been studied in [12,19]. The results in [12,19] rely on an uncertainty relation for pairs of general dictionaries and depend on the number of nonzero entries in $\mathbf{x}$ and $\mathbf{e}$, on the coherence parameters of the dictionaries $\mathbf{A}$ and $\mathbf{B}$, and on the amount of prior knowledge on the support of the signal and interference vector. However, the algorithms and proof techniques used in [12,19] cannot be adapted for the general (and practically more relevant) setting formulated in (1), which features approximately sparse signals and additive measurement noise.

1.2. Contributions

In this paper, we generalize the recovery guarantees of [12,19] to the framework (1) detailed above. In particular, we provide computationally efficient restoration and separation algorithms and derive corresponding recovery guarantees for the deterministic setting. Our guarantees depend in a natural way on the number of dominant nonzero entries of $\mathbf{x}$ and $\mathbf{e}$, on the coherence parameters of the dictionaries $\mathbf{A}$ and $\mathbf{B}$, and on the Euclidean norm of the measurement noise. Our results also depend on the amount of knowledge on the location of the dominant entries available prior to recovery. In particular, we investigate the following cases: (1) The locations of the dominant entries of the approximately sparse vector $\mathbf{x}$ and the support set of the perfectly sparse interference vector $\mathbf{e}$ are known (prior to recovery), (2) only the support set of the interference vector $\mathbf{e}$ is known, and (3) no support-set knowledge about $\mathbf{x}$ and $\mathbf{e}$ is available. Moreover, we present coherence-based bounds on the restricted isometry constants (RICs) for all these cases, which can be used to derive alternative recovery conditions. We provide a comparison to the recovery conditions for perfectly sparse signals and noiseless measurements presented in [12,19]. Finally, we demonstrate the efficacy of the proposed approach with two representative applications: restoration of audio signals impaired by a mixture of impulse noise and Gaussian noise, and removal of scratches from color photographs.

1.3. Notation

Lowercase and uppercase boldface letters stand for column vectors and matrices, respectively. The transpose, conjugate transpose, and (Moore–Penrose) pseudo-inverse of the matrix $\mathbf{M}$ are denoted by $\mathbf{M}^T$, $\mathbf{M}^H$, and $\mathbf{M}^\dagger = (\mathbf{M}^H\mathbf{M})^{-1}\mathbf{M}^H$, respectively. The kth entry of the vector $\mathbf{m}$ is $[\mathbf{m}]_k$, and the kth column of $\mathbf{M}$ is $\mathbf{m}_k$ and the entry in the kth row and ℓth column is designated by $[\mathbf{M}]_{k,\ell}$. The $M \times M$ identity matrix is denoted by $\mathbf{I}_M$ and the $M \times N$ all zeros matrix by $\mathbf{0}_{M\times N}$. The Euclidean (or ℓ_2) norm of the vector $\mathbf{x}$ is denoted by $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_1 = \sum_k |[\mathbf{x}]_k|$ stands for the ℓ_1-norm of $\mathbf{x}$, and $\|\mathbf{x}\|_0$