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# Fast thresholding algorithms with feedbacks for sparse signal recovery

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ABSTRACT

We provide another framework of iterative algorithms based on thresholding, feedback and null space tuning for sparse signal recovery arising in sparse representations and compressed sensing. Several thresholding algorithms with various feedbacks are derived. Convergence results are also provided. The core algorithm is shown to converge in finitely many steps under a (preconditioned) restricted isometry condition. The algorithms are seen as exceedingly effective and fast, particularly for large scale problems. Numerical studies about the effectiveness and the speed of the algorithms are also presented.


## 1. Introduction

A basic underdetermined linear inverse problem is about the solution to the system of linear equations

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times N}$ $(n \ll N)$ and $b \in \mathbb{R}^n$ are known. In the past few years, sparsity constraint has been a popular regularization approach toward the solution of such inverse problems. The problems of sparse representation and compressed sensing are typical examples.

The goal of sparse representation is to approximate a signal $b$ by a linear combination of the least number elementary signals/columns of (a dictionary) $A$, that is, to find the sparsest coefficient $x$ such that $Ax = b$. In compressed sensing, signals are assumed to be sparse in some transform domain. The ultimate goal is also to recover the sparse coefficient $x$ (and the signal) from a surprisingly small number of linear measurements of (fundamentally) the same form $Ax = b$. Evidently, the common problem here involves finding the sparsest solutions satisfying the linear equations. In other words, one wishes to solve an $\ell_0$-minimization problem

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$$(P_0) \qquad \min_{x\in\mathbb{R}^N} \|x\|_0 \quad \text{s.t.} \quad Ax = b,$$

where $\|\cdot\|_0$ is a quasi-norm standing for the number of the nonzero entries.

$(P_0)$ is clearly combinatorial in nature, and NP-hard in general [1]. The renowned advances in this area lie fundamentally in the replacement of $(P_0)$ with a convex relaxation

$$(P_1) \qquad \min_{x\in\mathbb{R}^N} \|x\|_1 \quad \text{s.t.} \quad Ax = b.$$

See a series of articles dealing with the equivalence between $(P_0)$ and $(P_1)$, e.g., [2–9]. Apparently, $(P_1)$ can also be solved by interior-point methods, such as [10,11] and a number of other different means.

Among others, "greedy algorithms" are another class of popular approaches of finding sparse solutions. Two typical representative approaches are Matching Pursuit (MP) and Orthogonal Matching Pursuit (OMP), e.g., [12–14]. In addition, a number of variants of the greedy pursuit algorithms have also been proposed by various authors, e.g., stagewise orthogonal matching pursuit (StOMP) [15], compressive sampling matching pursuit (CoSaMP) [16] and subspace pursuit (SP) [17], etc.

A third class of algorithms for sparse solutions to underdetermined linear inverse problems are iterative thresholding/shrinkage algorithms, which are known for their simplicity. Most iterative thresholding/shrinkage algorithms are motivated by minimizing a cost function, which combines a quadratic error term with a sparsity-promoting regularization term, for instance,

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Ax-b\|_2^2 + \lambda\|x\|_1.$$

Various iterative hard/soft thresholding algorithms [18–25], gradient-descent methods [26–28], and Bregman iterations [29,30] are representatives. Among this class of works, an algorithm [31] that replaces the $\ell_1$-regularization term by $\lambda\|x\|_0$ is proposed by Blumensath and Davies. An iterative hard thresholding (IHT) algorithm within the majorization minorization (MM) framework is analyzed [32]. It was also shown that IHT converges to a local minimum of the $\ell_0$-regularized cost function under some conditions.

In [33], Donoho and Maliki combine an exact solution to a small linear system with thresholding before and after the solution to derive a more sophisticated scheme, named two-stage thresholding (TST) method. Very recently, Foucart has proposed a hard thresholding pursuit (HTP) algorithm [34]. In essence, HTP can be regarded as a hybrid of IHT and CoSaMP.

In this article, a class of algorithms combining thresholding, feedbacks and null space tuning is proposed to find sparse solutions. The proposed algorithms are brought into a concise framework of *null space tuning* (NST). Several sparsity enhancing operators are incorporated into the NST framework to develop various algorithms. These algorithms are shown to be exceedingly fast and effective. Results about the theoretical performance and convergence are also presented.

The organization of this article is as follows. A brief description of the common framework of null space tuning is given in Section 2. The core algorithm, null space tuning with hard thresholding and feedback (NST + HT + FB), is introduced in Section 3. In Section 4, we present two other algorithms possessing the feedback nature, along with a brief study of the computational issues of the NST based algorithms. Section 5 is dedicated to the theoretical convergence studies of the NST + HT + FB algorithms. We show that the algorithm allows stable recovery of sparse vectors if the measurement matrix satisfies commonly known conditions. Extensive numerical tests and comparisons are presented in Section 6 to justify the advantages of the algorithms in practice.

## 2. A common framework of the approximation and null space tuning algorithms

Throughout this article, $A$ is commonly assumed to have full (row) rank. We propose the following iterative framework of the *approximation* and *null space tuning* (NST) algorithms

$$(\mathrm{NST}) \qquad \begin{cases} u^k = \mathbb{D}(x^k), \\ x^{k+1} = x^k + \mathbb{P}(u^k - x^k). \end{cases}$$

Here $\mathbb{D}(x^k)$ approximates the desired solution by various principles, and $\mathbb{P} := I - A^*(AA^*)^{-1}A$ is the orthogonal projection onto $\ker A$. The feasibility of $x^0$ is assumed, which guarantees that the sequence $\{x^k\}$ are all feasible. Obviously, $u^k \to x$ is expected as $k$ increases.

Due to the feasibility of the sequence $\{x^k\}$, the NST step can be rewritten as

$$\begin{aligned} x^{k+1} &= x^k + \mathbb{P}(u^k - x^k) \\ &= x^k + \left[I - A^*(AA^*)^{-1}A\right](u^k - x^k) \\ &= u^k + A^*(AA^*)^{-1}(b - Au^k), \end{aligned} \tag{2}$$

which indicates that $x^{k+1} - u^k$ is perpendicular to the hyperplane $\{x\colon\ Ax = b\}$. Therefore, $x^{k+1}$ is the orthogonal projection of $u^k$ onto the feasible set.