Appl. Comput. Harmon. Anal. 37 (2014) 106–125

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Saving phase: Injectivity and stability for phase retrieval

Afonso S. Bandeira [a], Jameson Cahill [b], Dustin G. Mixon [c,*], Aaron A. Nelson [c]

[a] Program in Applied and Computational Mathematics (PACM), Princeton University, Princeton, NJ 08544, United States
[b] Department of Mathematics, University of Missouri, Columbia, MO 65211, United States
[c] Department of Mathematics and Statistics, Air Force Institute of Technology, Wright-Patterson Air Force Base, OH 45433, United States

ARTICLE INFO

Article history:
Received 16 March 2013
Received in revised form 12 October 2013
Accepted 25 October 2013
Available online 29 October 2013
Communicated by Radu Balan

Keywords:
Phase retrieval
Quantum mechanics
Bilipschitz function
Cramer–Rao lower bound

ABSTRACT

Recent advances in convex optimization have led to new strides in the phase retrieval problem over finite-dimensional vector spaces. However, certain fundamental questions remain: What sorts of measurement vectors uniquely determine every signal up to a global phase factor, and how many are needed to do so? Furthermore, which measurement ensembles yield stability? This paper presents several results that address each of these questions. We begin by characterizing injectivity, and we identify that the complement property is indeed a necessary condition in the complex case. We then pose a conjecture that $4M-4$ generic measurement vectors are both necessary and sufficient for injectivity in M dimensions, and we prove this conjecture in the special cases where $M=2,3$. Next, we shift our attention to stability, both in the worst and average cases. Here, we characterize worst-case stability in the real case by introducing a numerical version of the complement property. This new property bears some resemblance to the restricted isometry property of compressed sensing and can be used to derive a sharp lower Lipschitz bound on the intensity measurement mapping. Localized frames are shown to lack this property (suggesting instability), whereas Gaussian random measurements are shown to satisfy this property with high probability. We conclude by presenting results that use a stochastic noise model in both the real and complex cases, and we leverage Cramer–Rao lower bounds to identify stability with stronger versions of the injectivity characterizations.

Published by Elsevier Inc.

1. Introduction

Signals are often passed through linear systems, and in some applications, only the pointwise absolute value of the output is available for analysis. For example, in high-power coherent diffractive imaging, this loss of phase information is eminent, as one only has access to the power spectrum of the desired signal [9]. *Phase retrieval* is the problem of recovering a signal from absolute values (squared) of linear measurements, called *intensity measurements*. Note that phase retrieval is often impossible—intensity measurements with the identity basis effectively discard the phase information of the signal's entries, and so this measurement process is not at all injective; the power spectrum similarly discards the phases of Fourier coefficients. This fact has led many researchers to invoke a priori knowledge of the desired signal, since intensity measurements might be injective when restricted to a smaller signal class. Unfortunately, this route has yet to produce practical phase retrieval guarantees, and practitioners currently resort to various ad hoc methods that often fail to work.

Thankfully, there is an alternative approach to phase retrieval, as introduced in 2006 by Balan, Casazza and Edidin [7]: Seek injectivity, not by finding a smaller signal class, but rather by designing a larger ensemble of intensity measurements. In [7], Balan et al. characterized injectivity in the real case and further leveraged algebraic geometry to show that $4M-2$ intensity measurements suffice for injectivity over M-dimensional complex signals. This realization that so few measurements

* Corresponding author.

1063-5203/$ – see front matter Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.acha.2013.10.002

can yield injectivity has since prompted a flurry of research in search of practical phase retrieval guarantees [2,4,6,12–14,18,19,37]. Notably, Candès, Strohmer and Voroninski [14] viewed intensity measurements as Hilbert–Schmidt inner products between rank-1 operators, and they applied certain intuition from compressed sensing to stably reconstruct the desired M-dimensional signal with semidefinite programming using only $\mathcal{O}(M \log M)$ random measurements; similar alternatives and refinements have since been identified [12,13,18,37]. Another alternative exploits the polarization identity to discern relative phases between certain intensity measurements; this method uses $\mathcal{O}(M \log M)$ random measurements in concert with an expander graph, and comes with a similar stability guarantee [2].

Despite these recent strides in phase retrieval algorithms, there remains a fundamental lack of understanding about what it takes for intensity measurements to be injective, let alone whether measurements yield stability (a more numerical notion of injectivity). For example, until very recently, it was believed that $3M - 2$ intensity measurements sufficed for injectivity (see for example [12]); this was disproved by Heinosaari, Mazzarella and Wolf [24], who used embedding theorems from differential geometry to establish the necessity of $(4+o(1))M$ measurements. As far as stability is concerned, the most noteworthy achievement to date is due to Eldar and Mendelson [19], who proved that $\mathcal{O}(M)$ Gaussian random measurements separate distant M-dimensional real signals with high probability. Still, the following problem remains wide open:

Problem 1. What are the necessary and sufficient conditions for measurement vectors to yield injective and stable intensity measurements?

The present paper addresses this problem in a number of ways. Section 2 focuses on injectivity, and it starts by providing the first known characterization of injectivity in the complex case (Theorem 4). Next, we make a rather surprising identification: that intensity measurements are injective in the complex case precisely when the corresponding phase-only measurements are injective in some sense (Theorem 5). We then use this identification to prove the necessity of the complement property for injectivity (Theorem 7). Later, we conjecture that $4M - 4$ intensity measurements are necessary and sufficient for injectivity in the complex case, and we prove this conjecture in the cases where $M = 2, 3$ (Theorems 10 and 12). Our proof for the $M = 3$ case leverages a new test for injectivity, which we then use to verify the injectivity of a certain quantum-mechanics-inspired measurement ensemble, thereby suggesting a new refinement of Wright's conjecture from [36] (see Conjecture 13).

We devote Section 3 to stability. Here, we start by focusing on the real case, for which we give upper and lower Lipschitz bounds of the intensity measurement mapping in terms of singular values of submatrices of the measurement ensemble (Lemma 16 and Theorem 18); this suggests a new matrix condition called the *strong complement property*, which strengthens the complement property of Balan et al. [7] and bears some resemblance to the restricted isometry property of compressed sensing [11]. As we will discuss, our result corroborates the intuition that localized frames fail to yield stability. We then show that Gaussian random measurements satisfy the strong complement property with high probability (Theorem 20), which nicely complements the results of Eldar and Mendelson [19]. In particular, we find an explicit, intuitive relation between the Lipschitz bounds and the number of intensity measurements per dimension (see Fig. 1). Finally, we present results in both the real and complex cases using a stochastic noise model, much like Balan did for the real case in [4]; here, we leverage Cramer–Rao lower bounds to identify stability with stronger versions of the injectivity characterizations (see Theorems 21 and 23).

1.1. Notation

Given a collection of measurement vectors $\Phi = \{\varphi_n\}_{n=1}^N$ in $V = \mathbb{R}^M$ or $\mathbb{C}^M$, consider the intensity measurement process defined by

$$\big(\mathcal{A}(x)\big)(n) := \big|\langle x, \varphi_n\rangle\big|^2.$$

Note that $\mathcal{A}(x) = \mathcal{A}(y)$ whenever $y = cx$ for some scalar c of unit modulus. As such, the mapping $\mathcal{A}: V \to \mathbb{R}^N$ is necessarily not injective. To resolve this (technical) issue, throughout this paper, we consider sets of the form V/S, where V is a vector space and S is a multiplicative subgroup of the field of scalars. By this notation, we mean to identify vectors $x, y \in V$ for which there exists a scalar $c \in S$ such that $y = cx$; we write $y \equiv x \bmod S$ to convey this identification. Most (but not all) of the time, V/S is either $\mathbb{R}^M/\{\pm 1\}$ or $\mathbb{C}^M/\mathbb{T}$ (here, $\mathbb{T}$ is the complex unit circle), and we view the intensity measurement process as a mapping $\mathcal{A}: V/S \to \mathbb{R}^N$; it is in this way that we will consider the measurement process to be injective or stable.

2. Injectivity

2.1. Injectivity and the complement property

Phase retrieval is impossible without injective intensity measurements. In their seminal work on phase retrieval [7], Balan, Casazza and Edidin introduce the following property to analyze injectivity:

Definition 2. We say $\Phi = \{\varphi_n\}_{n=1}^N$ in $\mathbb{R}^M$ ($\mathbb{C}^M$) satisfies the *complement property* (*CP*) if for every $S \subseteq \{1, \dots, N\}$, either $\{\varphi_n\}_{n\in S}$ or $\{\varphi_n\}_{n\in S^c}$ spans $\mathbb{R}^M$ ($\mathbb{C}^M$).