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Symmetric tight framelet filter banks with three high-pass filters ☆

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ABSTRACT

In this paper we continue our investigation of symmetric tight framelet filter banks (STFFBs) with a minimum number of generators in [6]. In particular, we shall systematically study STFFBs with three high-pass filters which are derived from the oblique extension principle. To our best knowledge, except the papers [1,10], there are no other papers in the literature so far systematically studying this problem. In this paper we show that there are two different types, called type I and type II, of STFFBs with three high-pass filters. Then we provide a detailed analysis and a complete algorithm to obtain all type I STFFBs with three high-pass filters. Our results not only significantly generalize the results in [1,10], but also help us answer several unresolved problems on STFFBs. Based on [6], we also propose an algorithm to construct all type II STFFBs with three high-pass filters and with the shortest possible filter supports. Several examples are given to illustrate the results and algorithms in this paper.

1. Introduction and motivations

Motivated by the interesting papers by Chui and He [1] and Han and Mo [10], continuing our lines developed in [6,8] on symmetric tight framelet filter banks with a minimum number of generators, in this paper we are particularly interested in systematically studying and developing algorithms to construct all symmetric tight framelet filter banks with three high-pass filters and with the shortest possible filter supports.

To proceed further, let us recall some definitions and notation. By $l_0(\mathbb{Z})$ we denote the linear space of all sequences $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ on $\mathbb{Z}$ such that $\{k \in \mathbb{Z}\colon\ u(k) \neq 0\}$ is a finite set. For $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$, its z-transform is a Laurent polynomial defined to be $\mathsf{u}(z) := \sum_{k\in\mathbb{Z}} u(k)z^k$. For a matrix $\mathsf{P}(z) = \sum_{k\in\mathbb{Z}} P_k z^k$ of Laurent polynomials, we define $\mathsf{P}^\star(z) := \sum_{k\in\mathbb{Z}} \overline{P_k}^{\mathsf{T}} z^{-k}$, where $\overline{P_k}^{\mathsf{T}}$ denotes the complex conjugate of the transpose of the matrix P_k.

The oblique extension principle introduced in [2,3] is a general procedure to construct tight wavelet frames through the design of tight framelet filter banks. Let $\Theta, a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$ with $\boldsymbol{\Theta}^\star = \boldsymbol{\Theta}$. We say that $\{a; b_1, \ldots, b_s\}_\Theta$ is *a tight framelet filter bank* if

$$\begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix} \begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix}^\star = \mathcal{M}_{a,\Theta}(z), \tag{1.1}$$

where

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$$\mathcal{M}_{a,\Theta}(z) := \begin{bmatrix} \boldsymbol{\Theta}(z) - \boldsymbol{\Theta}(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) & -\boldsymbol{\Theta}(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) \\ -\boldsymbol{\Theta}(z^2)\mathsf{a}(-z)\mathsf{a}^\star(z) & \boldsymbol{\Theta}(-z) - \boldsymbol{\Theta}(z^2)\mathsf{a}(-z)\mathsf{a}^\star(-z) \end{bmatrix}. \tag{1.2}$$

In particular we write $\{a; b_1, \dots, b_s\}$ for $\{a; b_1, \dots, b_s\}_{\boldsymbol{\delta}}$, where $\boldsymbol{\delta}$ is the *Dirac sequence* such that $\boldsymbol{\delta}(0) = 1$ and $\boldsymbol{\delta}(k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. Recall that a sequence $u : \mathbb{Z} \to \mathbb{C}$ has *symmetry* if

$$u(k) = \epsilon u(c - k), \quad \forall k \in \mathbb{Z} \text{ with } \epsilon \in \{-1, 1\},\ c \in \mathbb{Z}. \tag{1.3}$$

The filter u is symmetric if (1.3) holds with $\epsilon = 1$, and is antisymmetric if (1.3) holds with $\epsilon = -1$.

Note that (1.1) implies $\mathcal{M}^\star_{a,\Theta} = \mathcal{M}_{a,\Theta}$, from which we must have $\boldsymbol{\Theta}^\star = \boldsymbol{\Theta}$. Consequently, since $\boldsymbol{\Theta}^\star = \boldsymbol{\Theta}$, we see that Θ is symmetric if and only if Θ has real coefficients.

Since filters that we consider in this paper are not necessarily real-valued, there is another closely related but different notion of symmetry. We say that u has *complex symmetry* if

$$u(k) = \epsilon \overline{u(c - k)}, \quad \forall k \in \mathbb{Z} \text{ with } \epsilon \in \{-1, 1\},\ c \in \mathbb{Z}. \tag{1.4}$$

Obviously, for a real-valued sequence u, there is no difference between symmetry and complex symmetry.

For a given low-pass filter a and a moment correcting filter Θ, to obtain high-pass filters $b_1, \dots, b_s$ in a tight framelet filter bank, we have to factorize the given matrix $\mathcal{M}_{a,\Theta}$ in (1.2) so that (1.1) holds. To reduce computational complexity in the implementation of a tight framelet filter bank, we often prefer a small number s of high-pass filters. If $s = 1$, then we must have $\det(\mathcal{M}_{a,\Theta}(z)) = 0$ for all $z \in \mathbb{C}\backslash\{0\}$ which is too restrictive to be satisfied by many filters a and Θ. In fact, a tight framelet filter bank $\{a; b_1\}_\Theta$ with $s = 1$ is essentially an orthogonal wavelet filter bank, see [7, Theorem 7]. When $s = 2$, a necessary and sufficient condition has been given in [6, Theorem 4.2] (also see [8,11] for special cases) in terms of the filters a and Θ such that $\{a; b_1, b_2\}_\Theta$ is a tight framelet filter bank with [complex] symmetry. Moreover, several algorithms have been proposed in [6,8] to construct tight framelet filter banks $\{a; b_1, b_2\}_\Theta$ with [complex] symmetry. However, for any given low-pass filter a and a moment correcting filter Θ, the necessary and sufficient condition in [6] is still too restrictive. As a matter of fact, there are only a handful examples of symmetric tight framelet filter banks $\{a; b_1, b_2\}_\Theta$ with two high-pass filters known in the literature ([2,3,6,8,11–14] and references therein).

To have more flexibility in constructing tight framelet filter banks with [complex] symmetry from a given low-pass filter a and a moment correcting filter Θ, it is very natural to consider more than two high-pass filters. This naturally leads us to study in this paper symmetric tight framelet filter banks with three high-pass filters. For the particular case $s = 3$, the perfect reconstruction condition in (1.1) can be rewritten as

$$\boldsymbol{\Theta}(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) + \mathsf{b}_1(z)\mathsf{b}_1^\star(z) + \mathsf{b}_2(z)\mathsf{b}_2^\star(z) + \mathsf{b}_3(z)\mathsf{b}_3^\star(z) = \boldsymbol{\Theta}(z) \tag{1.5}$$

and

$$\boldsymbol{\Theta}(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) + \mathsf{b}_1(z)\mathsf{b}_1^\star(-z) + \mathsf{b}_2(z)\mathsf{b}_2^\star(-z) + \mathsf{b}_3(z)\mathsf{b}_3^\star(-z) = 0. \tag{1.6}$$

Currently, there are two particular constructions proposed in [1,10] for designing symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with particular choices of moment correcting filters Θ. For the special case $\Theta = \boldsymbol{\delta}$, Chui and He [1] found a simple solution for constructing a real-valued symmetric tight framelet filter bank $\{a; b_1, b_2, b_3\}$. More precisely, for any real-valued low-pass filter a having symmetry and satisfying

$$\mathsf{a}(z)\mathsf{a}^\star(z) + \mathsf{a}(-z)\mathsf{a}^\star(-z) \leqslant 1, \quad \forall z \in \mathbb{T} := \{\zeta \in \mathbb{C}\colon |\zeta| = 1\}, \tag{1.7}$$

define filters b_1, b_2, b_3 by (see [1, Proof of Theorem 3])

$$\mathsf{b}_1(z) := \big[\mathsf{u}(z^2) + z\mathsf{u}^\star(z^2)\big]/2, \qquad \mathsf{b}_2(z) := \big[\mathsf{u}(z^2) - z\mathsf{u}^\star(z^2)\big]/2, \qquad \mathsf{b}_3(z) := z\mathsf{a}^\star(-z), \tag{1.8}$$

where u is a Laurent polynomial with real coefficients obtained via the Fejér–Riesz lemma through

$$1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z) = \mathsf{u}(z^2)\mathsf{u}^\star(z^2). \tag{1.9}$$

Then it is straightforward to directly check that $\{a; b_1, b_2, b_3\}$ is a real-valued tight framelet filter bank with symmetry. Conversely, if $\{a; b_1, b_2, b_3\}$ is a tight framelet filter bank, then the condition in (1.7) on the filter a must hold [1]. Indeed, from the perfect reconstruction condition in (1.1), we must have $\det(\mathcal{M}_{a,\boldsymbol{\delta}}(z)) \geqslant 0$ for all $z \in \mathbb{T}$. Since $\det(\mathcal{M}_{a,\boldsymbol{\delta}}(z)) = 1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z)$, we see that (1.7) must hold.

We now describe the method in [10]. Let a be a real-valued low-pass filter with symmetry. Suppose that there exists a Laurent polynomial $\boldsymbol{\theta}$ with symmetry and real coefficients such that

$$\boldsymbol{\theta}^\star(-z)\boldsymbol{\theta}(z) = \boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z), \qquad \boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z) - \boldsymbol{\Theta}(z^2) \geqslant 0, \quad \forall z \in \mathbb{T}, \tag{1.10}$$

where

$$\boldsymbol{\Theta}(z) := \boldsymbol{\theta}^\star(z)\big[\mathsf{a}(z)\mathsf{a}^\star(z)\boldsymbol{\theta}(-z) + \mathsf{a}(-z)\mathsf{a}^\star(-z)\boldsymbol{\theta}(z)\big]. \tag{1.11}$$