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Letter to the Editor

An approximate sparsity model for inpainting ☆

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A B S T R A C T

Existing sparse inpainting models often suffer from their over-constraints on the sparsity of the transformed recovered images. Due to the fact that a transformed image of a wavelet or framelet transform is not truly sparse, but approximately sparse, we introduce an approximate sparsity model for inpainting. We formulate the model as minimizing the number of nonzero components of the soft-thresholding operator applied to the transformed image. The key difference of the proposed model from the existing ones is the use of a soft-thresholding operator which shrinkages the components of the transformed image. To efficiently solve the resulting nonconvex optimization problem, we rewrite the ℓ_0 norm, which counts the number of nonzero components, as a weighted ℓ_1 norm with a nonlinear discontinuous weight function, which is then approximated by a continuous weight function. We overcome the nonlinearity in the weight function by an iteration which leads to a numerical scheme for solving the nonconvex optimization problem. In each iteration, we solve a weighted ℓ_1 convex optimization problem. We then focus on understanding the existence of solutions of the weighted ℓ_1 convex optimization problem and characterizing them as fixed-points of a nonlinear mapping. The fixed-point formulation allows us to employ efficient iterative algorithms to find the fixed-points. Numerical experiments are shown to demonstrate improvement in performance of the proposed model over the existing models for image inpainting.

1. Introduction

In many applications, we often have an image with some pixels missing due to various reasons including impulsive noise caused by malfunctioning pixels in camera sensors, a scratch on text or signature in a picture. It is highly desirable to recover its original image from a damaged image. Image inpainting, first introduced to digital image processing in [1], is a mean to fulfill this goal. Specifically, image inpainting refers to the recovery of missing pixels in a digital image based on pixels available in the observed image. Many of its applications can be found in image processing, such as old films restoration [1], text or scratch removal [2,7], and cloud removal from remotely sensed images [28].

A variety of successful inpainting methods were proposed in the last decade. A group of classical image inpainting algorithms were developed based on numerical solutions of partial differential equations [1,2,13,14]. Algorithms of this kind propagate available information from an observed region to missing regions and experiments show that they perform well

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for piecewise smooth images with sharp edges. Another popular group of methods are path-based [4,22,34]. Algorithms in this category fill in missing pixels by exploring repetitions of local information and they perform well for missing regions with texture and of large sizes.

Recently, approaches based on sparse representation were applied to image inpainting [5,6,8,26]. More precisely, due to the assumption that images are piecewise smooth, these approaches assume that the images can be sparsely represented by some redundant systems, which are generated by a set of transforms such as wavelets, the discrete cosine transform, framelets and curvelets. In this setting, the inpainting problem is modeled as an optimization problem, whose objective function involves the ℓ_0 norm of the transformed coefficients of the underlying image with a redundancy system. However, the ℓ_0 norm is nonconvex and the associated optimization problem becomes a combinatorial problem which is NP-complete. A common alternative is to replace the ℓ_0 norm that appears in the cost functional by the ℓ_1 norm. As a result, it relaxes the nonconvex optimization problem to a convex one and therefore it allows us to develop fast algorithms for solving the convex optimization problem. The use of the ℓ_1 norm in sparse signal recovery has been well studied in the literature (see e.g. [9,10,21]).

Existing sparsity models often suffer from their over-constraints on the sparsity of the recovered images in the transformed domain. Outputs of a wavelet or framelet transform are often not truly sparse and instead, they are approximately sparse in the sense that most of the components of the vectors obtained from such a transform are not zero but have small absolute values compared to the remaining components. It is well known that a wavelet transform, a DCT transform or a framelet transform makes the energy of a transformed image concentrated on certain small regions. In other words, the transformed image is approximately sparse even though it is not truly sparse. This demands a model suitable for approximately sparse signals. The purpose of this paper is to introduce an approximate sparsity inpainting model which takes this important feature of such a transform into account. The model is expressed as the minimization of an objective function which is the ℓ_0 norm of the soft-thresholding operator applied to the transformed image over the constrained set described by the region having missing pixels. The key difference of the proposed model from the existing ones is the use of the soft-thresholding operator to truncate the components of the transformed image which are smaller than a threshold. The resulting optimization problem contains the ℓ_0 norm in its objective function and thus it is nonconvex. To solve the nonconvex optimization problem efficiently, we rewrite the ℓ_0 norm as a weighted ℓ_1 norm with a *nonlinear discontinuous* weight function. We then approximate the discontinuous weight function by a continuous one and use it to design an iteration scheme for solving the nonconvex optimization problem. At each step of the iteration scheme, we solve a weighted ℓ_1 convex optimization problem, which is the core minimization problem. Solutions of the core minimization problem are characterized as a system of fixed-point equations, for which efficient iterative algorithms were developed in the last few years [23–25,30,31]. Numerical experiments show that the proposed model outperforms the FBIA method in [8].

This paper is organized in four sections. In Section 2, we propose an approximate sparsity inpainting model and introduce a heuristic method to solve the resulting nonconvex optimization problem that describes the model. The method leads to an iteration scheme for solving the nonconvex problem, each step of whose iteration is to solve a core minimization problem. Section 3 is devoted to a study of the solvability of the core minimization problem and the fixed-point formulation of its solutions. In Section 4, we consider in details the algorithm for solving the approximate sparsity inpainting model. In Appendix A, numerical experiments are presented to demonstrate the accuracy of the proposed model in comparison with well-known existing sparse models.

2. Inpainting model based on approximate sparsity

In this section, we propose an approximate sparsity inpainting model and develop an efficient numerical algorithm for solving it.

We first present a generic form of an observation model for inpainting. Let $f \in \mathbb{R}^n$ be the original image defined on the domain $\Omega := \{1, 2, \ldots, n\}$. Let Λ be a proper subset of Ω and Λ^c be the complementary set of Λ in Ω. The observation model for an image g to be inpainted on Λ^c may be described as

$$g_i = \begin{cases} f_i, & \text{if } i \in \Lambda; \\ \text{arbitrary}, & \text{otherwise.} \end{cases} \tag{1}$$

This observation model says that the image g to be inpainted is identical to the original image f on set Λ. Therefore, an inpainted image for the given observation g and set Λ should be sought in the set

$$\mathcal{C} := \{f : f \in \mathbb{R}^n \text{ and } P_\Lambda f = P_\Lambda g\}, \tag{2}$$

where P_Λ is an $n \times n$ diagonal matrix whose i-th diagonal entry is 1 if $i \in \Lambda$ and 0 otherwise. The set $\mathcal{C}$ describes the constraints that the inpainted image must satisfy. This inpainting problem of inferring unavailable information on set Λ^c from available ones on set Λ is ill-posed. Therefore, it should be regularized based upon prior knowledge on the original image f.

An image is often assumed to be piecewise smooth. It can, therefore, have a sparse representation under a frame transform which has vanishing moments of a high order [29]. Furthermore, tight framelets are often employed since tight framelet transforms can be applied and inverted efficiently [15,17]. A discrete tight framelet is a matrix B such that $B^\top B = I$,