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On optimal wavelet reconstructions from Fourier samples: Linearity and universality of the stable sampling rate

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ABSTRACT

In this paper we study the problem of computing wavelet coefficients of compactly supported functions from their Fourier samples. For this, we use the recently introduced framework of generalized sampling. Our first result demonstrates that using generalized sampling one obtains a stable and accurate reconstruction, provided the number of Fourier samples grows linearly in the number of wavelet coefficients recovered. For the class of Daubechies wavelets we derive the exact constant of proportionality.
Our second result concerns the optimality of generalized sampling for this problem. Under some mild assumptions we show that generalized sampling cannot be outperformed in terms of approximation quality by more than a constant factor. Moreover, for the class of so-called perfect methods, any attempt to lower the sampling ratio below a certain critical threshold necessarily results in exponential ill-conditioning. Thus generalized sampling provides a nearly-optimal solution to this problem.

1. Introduction

One of the most fundamental problems in sampling theory is the issue of how to recover an object – an image or signal, for example – from a finite, and typically fixed, collection of its measurements. This problem lies at the heart of countless algorithms, with applications ranging from medical imaging to astronomy.

An important instance of this problem is the recovery of a compactly supported function from pointwise measurements of its Fourier transform. This problem occurs notably in Magnetic Resonance Imaging (MRI), as well as other applications such as radar. The classical approach for this problem is to recover f by computing a discrete Fourier transform (DFT) of the given data. However, this approach suffers from a number of drawbacks, including the sensitivity to motion and the presence of unpleasant Gibbs ringing [31,52]. Such phenomena can present serious issues in applications.

1.1. *Wavelets in imaging*

It is known that many real-life images can be much more efficiently represented by using wavelets than by their Fourier series. Images may be sparse in wavelets, or their coefficients may have improved decay properties. Representing medical images in this way also has several other benefits over the classical Fourier representation. These include better compressibility, improved feature detection (see [47,49] and references therein), and easier and more effective denoising [37,39,51].

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For these reasons, the use of wavelets in biomedical imaging applications has been a significant area of research for several decades [37,47,49].

Seeking to exploit these beneficial properties, an approach to recover wavelet coefficients directly in MRI was introduced in 1992 by Weaver et al. [31,52] (see also [28,37,41] and references therein). This is known as *wavelet-encoded* MRI. In this technique, the MR scanner itself is modified to sample wavelet coefficients along one dimension, with Fourier sampling, followed by a one-dimensional DFT, applied in the other. The resulting reconstructed image suffers less from Gibbs ringing, has fewer motion artefacts, and can in principle be acquired more rapidly [40,41]. For a medical perspective on wavelet encoding, and a discussion on how it can be combined with other imaging techniques such as parallel MRI, see [36].

Unfortunately, there are a number of disadvantages to wavelet encoding, which limit its applicability. These include low signal-to-noise ratio [40,52], and the extra complications encountered in the acquisition process due to having to modify the MR scanner [37]. Moreover, the state-of-the-art wavelet encoding allows only for reconstructions of wavelet coefficients of a 2D image in one direction, and thus does not permit one to take full advantage of general wavelets.

Nonetheless, the intensity of work on wavelets in MRI, and in particular on wavelet encoding techniques, indicates the importance of the problem of computing wavelet coefficients of biomedical imaging. It also serves to highlight the fact that this problem remains largely unsolved.

With this in mind, the purpose of this paper is to introduce and analyze a different solution to this problem, known as *generalized sampling*. Unlike wavelet encoding, which is primarily an engineering exercise in which the scanner itself is modified to produce different samples, we take the mathematical viewpoint and consider the samples as being fixed Fourier samples, and then seek to reconstruct wavelet coefficients directly via a post-processing algorithm. Our main conclusion is that one can perform wavelet encoding in applications such as MRI by generalized sampling without altering the scanner at all. This allows for the use of arbitrary wavelets and removes any hardware restrictions.

The typical MRI problem concerns the recovery of two- or three-dimensional images from Fourier measurements. In this paper, we shall consider only the one-dimensional case. As we explain further in Section 9, both the technique of generalized sampling and its analysis can be extended to the higher-dimensional setting. This is a topic of ongoing work. The development and analysis of the one-dimensional case, i.e. the topic of this paper, can be viewed as a vital first step in this direction.

Remark 1.1. The reader may wonder at this stage why wavelet encoding is necessary. Why could one not simply recover wavelet coefficients from standard MRI data by applying the DFT and DWT (discrete wavelet transform) in turn? There are two reasons. First, the use of DFT yields a discrete (pixel-based) version of the truncated Fourier series. Hence, by applying the DWT one (at best) obtains the wavelet coefficients of the truncated Fourier series and not the actual wavelet coefficients of the image itself. Second, the recovery algorithm using DFT and DWT would be as follows. The "wavelet coefficients" are obtained by

$$x = \mathrm{DWT} \cdot \mathrm{DFT}^{-1} y,$$

where y is a vector of the Fourier samples. However, when mapping these coefficients back to the pixel domain, one gets

$$\tilde{x} = \mathrm{DWT}^{-1} x = \mathrm{DFT}^{-1} y,$$

which is exactly what we would get in the first place using DFT. In particular, nothing is gained here in terms of the quality of the reconstructed image. By contrast, wavelet encoding techniques seek to reconstruct the true wavelet coefficients directly. This yields a different reconstruction with qualities determined by the wavelet used, and not by the original Fourier series.

1.2. Generalized sampling

In sampling theory, the mathematical problem of recovering the coefficients of a signal or image in a particular basis from samples taken with respect to another basis has been studied for several decades [45]. Motivating this is the fact that many images and signals can be better represented in terms of a different basis (e.g. splines [44] or the aforementioned wavelets) than the basis in which they are sampled (e.g. the Fourier basis). Some of the earliest work on this problem in its abstract form was carried out by Unser and Aldroubi, who introduced a mathematical reconstruction framework known as *consistent reconstructions* for shift-invariant sampling and reconstruction spaces [46] (see also [50]). This was later considered by Eldar et al. who extended this framework to frames in arbitrary Hilbert spaces [18–20,24]. Further developments to more general types of signal models were introduced in [38] (see also [7,21]).

Whilst consistent reconstructions are quite popular in engineering applications, there are a number of issues. As discussed in [2,3,22,33], consistent reconstructions have the significant drawback of being, in general, neither numerically stable nor convergent as the number of samples is increased. Hence, when applied to the important problem of recovering wavelet coefficients of MR images, they can result in severe amplification of noise and round-off error.

Nonetheless, it transpires that these issues can be overcome completely by using a different approach, known as *generalized sampling*. Introduced by Adcock and Hansen in [3,4], based on elements from [30], this framework allows one to recover a signal f modeled as an element of a separable Hilbert space $\mathcal{H}$ in terms of any Riesz basis $\{\varphi_j\}_{j=1}^{\infty}$ from samples