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Wavelet optimal estimations for a density with some additive noises

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ABSTRACT

Using wavelet methods, Fan and Koo study optimal estimations for a density with some additive noises over a Besov ball $B^s_{r,q}(L)$ $(r, q \geqslant 1)$ and over L^2 risk (Fan and Koo, 2002 [13]). The L^∞ risk estimations are investigated by Lounici and Nickl (2011) [19]. This paper deals with optimal estimations over L^p $(1 \leqslant p \leqslant \infty)$ risk for moderately ill-posed noises. A lower bound of L^p risk is firstly provided, which generalizes Fan–Koo and Lounici–Nickl's theorems; then we define a linear and non-linear wavelet estimators, motivated by Fan–Koo and Pensky–Vidakovic's work. The linear one is rate optimal for $r \geqslant p$, and the non-linear estimator attains suboptimal (optimal up to a logarithmic factor). These results can be considered as an extension of some theorems of Donoho et al. (1996) [10]. In addition, our non-linear wavelet estimator is adaptive to the indices s, r, q and L.

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1. Introduction and preliminary

The density estimation for a statistical model with additive noise plays important roles in both statistics and econometrics [17]. More precisely, let $Y_1, Y_2, \dots, Y_n$ be independent and identically distributed (i.i.d.) random variables of

$$Y = X + \epsilon, \tag{1.1}$$

where X stands for real-valued random variable with unknown probability density $f : \mathbb{R} \to \mathbb{R}^+$ and ϵ denotes an independent random noise (error) with the probability density φ. The problem is to estimate f by $Y_1, Y_2, \dots, Y_n$ in some sense.

As a deconvolution problem, the density g of Y equals to the convolution of f and φ. In particular, (1.1) reduces to the classical model with no errors, when φ degenerates to the Dirac functional δ $(g = f * \delta = f$ in that case). The traditional kernel method deals with that problem effectively [1,23,24]. However, it has two disadvantages: the first is the complexity of band choice for some densities; the second one: as a linear estimation, it doesn't give optimal convergence rates in many cases.

Another classical method, the Fourier based deconvolution, turns out to be effective for periodic densities under super smooth noises [4,11]. However, the Fourier system $\{e^{int}, n \in \mathbb{Z}\}$ is orthogonal in $L^2[0, 2\pi]$, it can't deal with aperiodic cases. Wavelets can, because a wavelet system constitutes an orthonormal basis of $L^2(\mathbb{R})$. Furthermore, a non-linear wavelet estimator (defined by thresholding) gives a better estimation than the classical methods, due to time and frequency localization of wavelet bases [10,13]. In addition, wavelets provide fast algorithm, which is important in numerical computations.

In 1996, Delyon and Juditsky [6] investigated the density estimation (without error) by compactly supported wavelets. Pensky and Vidakovic, Walter [20,25] studied Meyer wavelet estimation for densities in Sobolev space $W^s_2(\mathbb{R})$ in 1999; three

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years later, Fan and Koo considered wavelet estimation over L^2 risk and Besov space $B^s_{r,q}(\mathbb{R})$ with $1 \leqslant r \leqslant 2$ [13]. In 2011, Lounici and Nickl investigated optimal estimation over $B^s_{\infty,\infty}(\mathbb{R})$ and L^∞ risk by wavelet method [19].

This paper deals with L^p $(1 \leqslant p \leqslant \infty)$ risk estimation, which includes the important cases L^1, L^2 and L^∞ risk estimations in $B^s_{r,q}(\mathbb{R})$ $(q, r \in [1,\infty])$ for moderately ill-posed noises by using wavelet bases. Section 1.1 introduces some notations and classical results on wavelets and Besov spaces, which will be used in our discussions; the main results are presented in Section 1.2. We shall discuss some relations to the work in [10,13,19,20,25]. In order to prove our theorems in the last two sections, we show several lemmas in Section 2.

1.1. Some preparations

We begin with the concept of multiresolution analysis (MRA, [5]), which is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of the square integrable function space $L^2(\mathbb{R})$ satisfying the following properties:

(i) $V_j \subseteq V_{j+1}$, $\forall j \in \mathbb{Z}$. Here and after, $\mathbb{Z}$ denotes the integer set and $\mathbb{N} := \{n \in \mathbb{Z},\ n \geqslant 0\}$;
(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;
(iii) There exists $\phi(x) \in L^2(\mathbb{R})$ (scaling function) such that $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal system and $V_0 = \overline{span}\{\phi(x-k)\}$.

With the standard notation $h_{jk}(x) := 2^{\frac{j}{2}} h(2^j x - k)$ in wavelet analysis, we can derive a corresponding wavelet (function)

$$\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{h_{1-k}} \phi_{1k}(x) \quad \text{with } h_k = \langle \phi, \phi_{1k} \rangle$$

such that for a fixed $j \in \mathbb{Z}$, $\{\psi_{jk}(x)\}_{k\in\mathbb{Z}}$ constitutes an orthonormal basis of the orthogonal complement W_j of V_j in V_{j+1}. Moreover, for fixed $J \in \mathbb{N}$, both $\{\phi_{Jk}(x),\ \psi_{jk}(x)\}_{j\geqslant J, k\in\mathbb{Z}}$ and $\{\psi_{jk}(x)\}_{j,k\in\mathbb{Z}}$ are orthonormal bases of $L^2(\mathbb{R})$ [5]. Then each $f \in L^2(\mathbb{R})$ has two expansions in $L^2(\mathbb{R})$ sense

$$f = \sum_{k\in\mathbb{Z}} \alpha_{Jk}\phi_{Jk} + \sum_{j\geqslant J}\sum_{k\in\mathbb{Z}} \beta_{jk}\psi_{jk} \quad \text{and} \quad f = \sum_{j,k\in\mathbb{Z}} \beta_{jk}\psi_{jk}$$

with $\alpha_{jk} := \langle f, \phi_{jk}\rangle$ and $\beta_{jk} := \langle f, \psi_{jk}\rangle$.

As usual, let P_j and Q_j be the orthogonal projections from $L^2(\mathbb{R})$ to V_j and W_j respectively,

$$P_j f = \sum_{k\in\mathbb{Z}} \alpha_{jk}\phi_{jk}, \qquad Q_j f = \sum_{k\in\mathbb{Z}} \beta_{jk}\psi_{jk} = (P_{j+1} - P_j) f.$$

Then $f = P_J f + \sum_{j=J}^{\infty} Q_j f$.

Two important examples are Meyer and Daubechies' wavelets. To introduce Meyer wavelets, we need the Fourier transform of $f \in L(\mathbb{R})$,

$$\hat{f}(t) := (Ff)(t) = \int_{\mathbb{R}} f(x) e^{-itx}\, dx.$$

The classical method extends that definition to $L^2(\mathbb{R})$ functions (as a matter of fact, we can define the Fourier transform $\hat{f}$ of a generalized function f as well. In particular, when δ is the Dirac functional, $\hat{\delta} = 1$). A Meyer wavelet ψ satisfies $\hat{\psi} \in C^\infty(\mathbb{R})$ and the support of $\hat{\psi}$ contained in $\{t\colon \frac{2\pi}{3} \leqslant |t| \leqslant \frac{8\pi}{3}\}$. Daubechies wavelets D_{2N} $(N = 1, 2, \ldots)$ are compactly supported in time domain. They can be smooth enough with increasing supports as N gets large, although D_{2N} don't have analytic formulae except for $N = 1$.

The following simple lemma is fundamental in our discussions. We use $\|f\|_p$ to denote $L^p(\mathbb{R})$ norm for $f \in L^p(\mathbb{R})$, and $\|\lambda\|_p$ does $l^p(\mathbb{Z})$ norm for $\lambda \in l^p(\mathbb{Z})$, where

$$l^p(\mathbb{Z}) := \begin{cases} \{\lambda = \{\lambda_k\},\ \sum_{k\in\mathbb{Z}} |\lambda_k|^p < \infty\}, & p < \infty; \\ \{\lambda = \{\lambda_k\},\ \sup_{k\in\mathbb{Z}} |\lambda_k| < \infty\}, & p = \infty. \end{cases}$$

Lemma 1.1. *(See [14].) Let h be a scaling function or a wavelet with $\theta(h) := \sup_{x\in\mathbb{R}} \sum_k |h(x-k)| < \infty$. Then, there exist $C_2 \geqslant C_1 > 0$ such that for $\lambda = \{\lambda_k\} \in l^p(\mathbb{Z})$ and $1 \leqslant p \leqslant \infty$,*

$$C_1 2^{j(\frac{1}{2}-\frac{1}{p})} \|\lambda\|_p \leqslant \left\| \sum_{k\in\mathbb{Z}} \lambda_k h_{jk} \right\|_p \leqslant C_2 2^{j(\frac{1}{2}-\frac{1}{p})} \|\lambda\|_p.$$

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