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Wavelet transform on the torus: A group theoretical approach

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A B S T R A C T

We construct a continuous wavelet transform (CWT) on the torus $\mathbb{T}^2$ following a group-theoretical approach based on the conformal group $SO(2,2)$. The Euclidean limit reproduces wavelets on the plane $\mathbb{R}^2$ with two dilations, which can be defined through the natural tensor product representation of usual wavelets on $\mathbb{R}$. Restricting ourselves to a single dilation imposes severe conditions for the mother wavelet that can be overcome by adding extra modular group $SL(2,\mathbb{Z})$ transformations, thus leading to the concept of modular wavelets. We define modular-admissible functions and prove frame conditions.

1. Introduction

Since the pioneer work of Grossmann, Morlet and Paul [1], several extensions of the standard continuous wavelet transform (CWT) on $\mathbb{R}$ to general manifolds $\mathbb{X}$ have been constructed (see e.g. [2,3] for general reviews and [4,5] for recent papers on WT and Gabor systems on homogeneous manifolds). Particular interesting examples are the construction of CWT on: spheres $\mathbb{S}^{N-1}$, by means of an appropriate unitary representation of the Lorentz group in $N+1$ dimensions $SO(N,1)$ [6–10], on the upper sheet $\mathbb{H}^2_+$ of the two-sheeted hyperboloid $\mathbb{H}^2$ [11], or its stereographical projection onto the open unit disk $D_1 = SO(1,2)/SO(2)$, and the construction of conformal wavelets in the (compactified) complex Minkowski space [12]. The basic ingredient in all these constructions is a group of transformations G which contains dilations and motions on $\mathbb{X}$, together with a transitive action of G on $\mathbb{X}$.

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In this article we first extend the group theoretical construction of wavelets on the circle $\mathbb{S}^1$ based on the group $SL(2,\mathbb{R})$, given in [16], to wavelets on the two-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ based on the group $SO(2,2)$, and introduce additional modular transformations in $SL(2,\mathbb{Z})$, which lead to the concept of *modular wavelets*.

We must stress that the topological torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ can be obtained from the plane $\mathbb{R}^2$ by imposing periodic boundary conditions and these are often used in physical and mathematical models to simulate a large system by modeling a small part that is far from its edge. For instance, in the Quantum Hall Effect [13], the topology of the problem is that of a torus [14], and modular transformations are of crucial importance for the classification of fractional quantum numbers [15]. Moreover, the Discrete Fourier Transform, either in one or more dimensions, implicitly assumes that the signal or image is periodic, and this is a valid approximation as long as edge effects are negligible. Besides, wavelets on $\mathbb{R}^2$ (or higher dimensions) encounters applications in microlocal analysis [17], and thus wavelets on the torus would be helpful in toroidal microlocal analysis [18].

The organization of the paper is as follows. In Section 2 we briefly remind the group theoretical construction of the CWT on $\mathbb{S}^2$ based on the Lorentz group $SO(3,1)$, which serves as an introduction and to set notation. In Section 3 we construct the CWT on the topological torus $\mathbb{T}^2$ based on the group $SO(2,2)$, introducing admissibility conditions and proving the existence of admissible functions and continuous wavelets frames. This construction naturally relies on two dilations. Usual wavelet constructions rely on a single dilation but, in our construction, the frame property is lost when restricting to a single (let us say, diagonal) dilation. The way out is to introduce additional ingredients in the wavelet parameter space, like *modular transformations*, which lead to the concept of *modular wavelets*. This construction is made in Section 4.

2. CWT on the sphere $\mathbb{S}^2$ based on $SO(3,1)$: A reminder

Let us denote by $L^2(\mathbb{S}^2, d\Omega)$ the Hilbert space of square integrable functions on the two-sphere $\mathbb{S}^2$, with the usual measure $d\Omega = \sin\theta\, d\theta\, d\varphi$ (we shall omit $d\Omega$ and just write $L^2(\mathbb{S}^2)$). An orthonormal basis of $L^2(\mathbb{S}^2)$ is given in terms of spherical harmonics:

$$Y_l^m(\theta,\varphi) = N_{lm} P_l^m(\cos\theta) e^{im\varphi}, \quad l = 0,1,\ldots,\ m = -l,\ldots,l \tag{1}$$

fulfilling

$$\langle Y_l^m \mid Y_{l'}^{m'} \rangle = \int_{\theta=0}^{\pi} \int_{\varphi=-\pi}^{\pi} \overline{Y_l^m(\theta,\varphi)} Y_{l'}^{m'}(\theta,\varphi)\, d\Omega = \delta_{ll'}\delta_{mm'}, \tag{2}$$

with a convenient choice of normalization factors N_{lm}, where P_l^m are the associated Legendre polynomials.

The problem of defining a satisfactory dilation on the sphere was solved by Antoine and Vandergheynst in [7], where they used a group-theoretical approach based on the Lorentz group $G = SO(3,1)$. Dilation is embedded into G via the Iwasawa decomposition $G = KAN$ with K compact, A Abelian and N nilpotent subgroups. The parameter space X of their CWT is the quotient G/N. The expression for the dilation, with parameter $a > 0$, of the colatitude angle θ is

$$\theta_a = 2\arctan\big(a\tan(\theta/2)\big), \tag{3}$$

and it has a direct geometrical interpretation as a dilation around the North Pole of the sphere, lifted from the tangent plane by inverse stereographic projection. For any function $f \in L^2(\mathbb{S}^2)$, a unitary representation of this dilation is given by

$$\big[D_a^{\mathbb{S}^2} f\big](\theta,\varphi) = \lambda(a,\theta)^{1/2} f(\theta_{1/a},\varphi), \tag{4}$$