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Letter to the Editor

Invariance properties of generalized polarization tensors and design of shape descriptors in three dimensions ☆

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ABSTRACT

We derive transformation formulas for the generalized polarization tensors under rigid motions and scaling in three dimensions, and use them to construct an infinite number of invariants under those transformations. These invariants can be used as shape descriptors for dictionary matching.

1. Introduction

The shape of a domain can be represented in terms of various physical and geometric quantities such as eigenvalues, capacity and moments. The generalized polarization tensor (GPT) is one of them. GPTs are an (infinite) sequence of tensors associated with inclusions (domains) and they appear naturally in the far field expansion of the perturbation of the electrical field in the presence of the inclusion. They are geometric quantities in the sense that the full set of GPTs completely determines the shape of the inclusion. Moreover, one can use a first few terms of GPTs to recover a good approximation of the actual shape of an inclusion [3,4].

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When the domain is transformed by a rigid motion and a scaling, the corresponding GPTs change according to certain rules and it is possible to construct as combinations of GPTs invariants under these transformations. This property makes GPTs suitable for the dictionary matching problem. The dictionary matching problem is to identify the object in the dictionary when the target object is identical to one of the objects in the dictionary up to shifting, rotation, and scaling. The standard method of dictionary matching is to construct invariants, called shape descriptors, under rigid motions and scaling, and to compare those invariants. In [1], invariants are constructed using GPTs in two dimensions.

It is the purpose of this letter to extend results of [1] to construct invariants using GPTs in three dimensions. In fact, using contracted GPTs (CGPT), which are harmonic combinations of GPTs, we are able to construct an infinite number of shape descriptors. Using transformation formulas of spherical harmonics under rigid motions [5], we are able to derive transformation formulas obeyed by the CGPTs. We then use these formulas and the method for registration to construct invariants which can be used for target shape description.

2. The CGPT block matrix

For $D \Subset \mathbb{R}^3$ with Lipschitz boundary, the Neumann–Poincaré operator $\mathcal{K}_D^*$ is defined by

$$\mathcal{K}_D^*[\phi](x) = \frac{1}{4\pi}\text{p.v.}\int_{\partial D} \frac{\langle x-y, \nu(x)\rangle}{|x-y|^3}\phi(y)d\sigma(y), \quad x \in \partial D.$$

Here $\langle \cdot,\cdot \rangle$ denotes the scalar product and $\nu(x)$ the unit outward normal vector $x \in \partial D$. Let λ be a real number such that $|\lambda| > 1/2$. The generalized polarization tensor (GPT) $M_{\alpha\beta}$ for multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ associated with λ and D is defined by

$$M_{\alpha\beta}(\lambda, D) = \int_{\partial D} y^\beta \left(\lambda I - \mathcal{K}_D^*\right)^{-1}\left[\nu \cdot \nabla y^\alpha\right] d\sigma.$$

Here $y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$. Throughout this letter λ is fixed, so we use $M_{\alpha\beta}(D)$ for $M_{\alpha\beta}(\lambda, D)$.

The contracted GPTs (CGPT) are harmonic combinations of GPTs. Let Y_n^m, $-n \le m \le n$, be the (complex) spherical harmonic of homogeneous degree n and order m. We define the CGPT M_{nmkl} by

$$M_{nmlk}(D) = \int_{\partial D} r_y^l \overline{Y_l^k(\theta_y, \varphi_y)} \left(\lambda I - \mathcal{K}_D^*\right)^{-1}\left[\frac{\partial}{\partial \nu} r_y^n Y_n^m(\theta_y, \varphi_y)\bigg|_{\partial D}\right](y) d\sigma(y) \tag{2.1}$$

with $y = r_y(\cos\varphi_y \sin\theta_y, \sin\varphi_y \sin\theta_y, \cos\theta_y)$.

We introduce the matrix $\mathbf{M}_{ln}$ of dimension $(2l+1) \times (2n+1)$ by

$$(\mathbf{M}_{ln})_{km} := M_{nmlk}, \quad -l \le k \le l,\ -n \le m \le n. \tag{2.2}$$

For an integer K, we define

$$\mathbf{M} := \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1K} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \cdots & \mathbf{M}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{K1} & \mathbf{M}_{K2} & \cdots & \mathbf{M}_{KK} \end{bmatrix}.$$

We call $\mathbf{M}_{ln}$ and $\mathbf{M}$ the CGPT matrix and the CGPT block matrix of order K, respectively. The following proposition holds [2].