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Exponential spectra in $L^2(\mu)$ ☆

Xing-Gang He [a], Chun-Kit Lai [b,*], Ka-Sing Lau [b]

[a] *College of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China*
[b] *Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong*

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ABSTRACT

Let μ be a Borel probability measure with compact support. We consider exponential type orthonormal bases, Riesz bases and frames in $L^2(\mu)$. We show that if $L^2(\mu)$ admits an exponential frame, then μ must be of pure type. We also classify various μ that admits either kind of exponential bases, in particular, the discrete measures and their connection with integer tiles. By using this and convolution, we construct a class of singularly continuous measures that has an exponential Riesz basis but no exponential orthonormal basis. It is the first of such kind of examples.

1. Introduction

Throughout the paper we assume that μ is a (Borel) probability measure on $\mathbb{R}^d$ with compact support. We call a family $E(\Lambda) = \{e^{2\pi i\lambda x}\colon \lambda \in \Lambda\}$ (Λ is a countable set) a *Fourier frame* of the Hilbert space $L^2(\mu)$ if there exist $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{\lambda\in\Lambda} \left|\left\langle f, e^{2\pi i\lambda x}\right\rangle\right|^2 \leqslant B\|f\|^2, \quad \forall f \in L^2(\mu). \tag{1.1}$$

Here the inner product is defined as usual,

$$\left\langle f, e^{2\pi i\lambda x}\right\rangle = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\lambda x}\, d\mu(x).$$

$E(\Lambda)$ is called an (*exponential*) *Riesz basis* if it is both a basis and a frame of $L^2(\mu)$. Fourier frames and exponential Riesz bases are natural generalizations of exponential orthonormal bases in $L^2(\mu)$. They have fundamental importance in non-harmonic Fourier analysis and close connection with time-frequency analysis [2,8,9]. When (1.1) is satisfied, $f \in L^2(\mu)$ can be expressed as $f(x) = \sum_{\lambda\in\Lambda} c_\lambda e^{2\pi i\lambda x}$, and the expression is unique if it is a Riesz basis.

When $E(\Lambda)$ is an orthonormal basis (Riesz basis, or frame) of $L^2(\mu)$, we say that μ is a *spectral measure* (*R-spectral measure, or F-spectral measure respectively*) and Λ is called a *spectrum* (*R-spectrum, or F-spectrum respectively*) of $L^2(\mu)$. We will also use the term orthonormal spectrum instead of spectrum when we need to emphasis the orthonormal property. If

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* Corresponding author.
E-mail addresses: xingganghe@yahoo.com.cn (X.-G. He), cklai@math.cuhk.edu.hk (C.-K. Lai), kslau@math.cuhk.edu.hk (K.-S. Lau).

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$E(\Lambda)$ only satisfies the upper bound condition in (1.1), then it is called a *Bessel set* (or *Bessel sequence*); for convenience, we also call Λ a Bessel set of $L^2(\mu)$.

One of the interesting and basic questions in non-harmonic Fourier analysis is:

What kind of compactly supported probability measures in $\mathbb{R}^d$ belong to the above classes of measures?

When μ is the restriction of the Lebesgue measure on K with positive measure, the question whether it is a spectral measure is related to the well known Fuglede problem of translational tiles (see [7,22,14,26] and the reference therein). While it is easy to show that such μ is an F-measure, it is an open question whether it is an R-spectral measure. If K is a unit interval, its F-spectrum was completely classified in terms of de Brange's theory of entire functions [23]. In another general situation, Lai [13] proved a sharp result that if μ is absolutely continuous with respect to the Lebesgue measure, then it is an F-spectral measure if and only if its density function is essentially bounded above and below on the support.

The problem becomes more intriguing when μ is singular. The first example of such spectral measures was given by Jorgensen and Pedersen [11]. They showed that the Cantor measures with even contraction ratio ($\rho = 1/2k$) is spectral, but the one with odd contraction ratio ($\rho = 1/(2k+1)$) is not. This raises the very interesting question on the existence of an exponential Riesz basis or a Fourier frame for such measures, and more generally for the self-similar measures [15,16,6,25,10]. In particular Dutkay et al. proposed to use the *Beurling dimension* as some general criteria for the existence of Fourier frame [4]. They also attempted to find a self-similar measure which admits an exponential Riesz basis or a Fourier frame but not an exponential orthonormal basis [5]. However, no such examples have been found up to now.

In this paper, we will carry out a detail study of the three classes of spectra mentioned. It is known that a spectral measure must be either purely discrete or purely continuous [16]. Our first theorem is a pure type law for the F-spectral measures.

Theorem 1.1. *Let μ be an F-spectral measure on $\mathbb{R}^d$. Then it must be one of the three pure types: discrete (and finite), singularly continuous or absolutely continuous.*

For the proof, the discrete case is based on the frame inequality, and the two continuous cases make use the concept of *lower Beurling density* of the F-spectrum.

To complete the previous digression on the continuous measures, we have the following conclusions for finite discrete measures.

Theorem 1.2. *Let $\mu = \sum_{c\in C} p_c\delta_c$ be a discrete probability measure in $\mathbb{R}^d$ with $\mathcal{C}$ a finite set. Then μ is an R-spectral measure.*

To determine such discrete μ to be a spectral measure, we will restrict our consideration on $\mathbb{R}^1$ and let $\mathcal{C} \subset \mathbb{Z}^+$ with $0 \in \mathcal{C}$. Then the Fourier transform of μ is

$$\widehat{\mu}(x) = p_0 + p_1 e^{2\pi i c_1} x + \cdots + p_{k-1} e^{2\pi i c_{k-1} x} := m_\mu(x),$$

where $P = \{p_i\}_{i=0}^{k-1}$ is a set of probability weights. We call $m_\mu(x)$ the *mask polynomial* of μ. Let $\mathcal{Z}_\mu = \{x \in [0,1)\colon\ m_\mu(x) = 0\}$ be the zero set of $m_\mu(x)$, and Λ is called a *bi-zero set* if $\Lambda - \Lambda \subset \mathcal{Z}_\mu \cup \{0\}$. Denote the cardinality of E by $\#E$. It is easy to see the following simple proposition.

Proposition 1.3. *Let $\mu = \sum_{c\in\mathcal{C}} p_c\delta_c$ with $\mathcal{C} \in \mathbb{Z}^+$ and $0 \in \mathcal{C}$. Then μ is a spectral measure if and only if there is a bi-zero set Λ of m_μ and $\#\Lambda = \#\mathcal{C}$. In this case, all the p_c are equal.*

The determination of the bi-zero set is, however, non-trivial, as the zeros of a mask polynomial is rather hard to handle. As an implementation of the proposition, we work out explicit expressions of the set $\mathcal{C}$ and the bi-zero set when $\#\mathcal{C} = 3, 4$. It is difficult to have such expression beyond 4 directly. On the other hand, there are systematic studies of the zeros of the mask polynomials by factorizing the mask polynomial as cyclotomic polynomials (the minimal polynomial of the root of unity). This has been used to study the integer tiles and their spectra (see [3,14,19]). We adopt this approach to a class of self-similar measures (which is continuous) in our consideration:

Let $n > 0$ and let $\mathcal{A} \subset \mathbb{Z}^+$ be a finite set with $0 \in \mathcal{A}$, we define a self-similar measure $\mu := \mu_{\mathcal{A},n}$ by

$$\mu(E) = \frac{1}{\#A} \sum_{a\in\mathcal{A}} \mu(nE - a)$$

where E is a Borel subset in $\mathbb{R}$. Note that the Lebesgue measure on $[0,1]$ and the Cantor measures are such kind of measures. The following theorem is a combination of the results in [24,14] and [15]:

Theorem 1.4. *Let $\mathcal{A} \subset \mathbb{Z}^+$ be a finite set with $0 \in \mathcal{A}$. Suppose there exists $\mathcal{B} \subset \mathbb{Z}^+$ such that $\mathcal{A} \oplus \mathcal{B} = \mathbb{N}_n$ where $\mathbb{N}_n = \{0,\dots,n-1\}$. Then $\delta_{\mathcal{A}} = \sum_{a\in\mathcal{A}} \delta_a$ is a spectral measure with a spectrum in $\frac{1}{n}\mathbb{Z}$; the associated self-similar measure $\mu_{\mathcal{A},n}$ is also a spectral measure, and it has a spectrum in $\mathbb{Z}$ if $\gcd\mathcal{A} = 1$.*