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Case Studies

Signal processing by alternate dual Gabor frames

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Duality principles in Gabor frame theory have a key roll in applications. Each dual frame enable us to write every element in the underlying Hilbert space as a linear combination of the frame elements. Canonical dual is used usually in this construction. In this article, it is proved that by using alternate duals we obtain more accurate results.

1. Introduction and preliminaries

Gabor frames play a very important role in signal analysis and many other parts of applied mathematics [1,9,11]. They are generated by modulations and translations of one single function. A Gabor frame is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for some $g \in L^2(\mathbb{R})$ and $a, b > 0$, where $T_{na}f(x) = f(x - na)$ and $E_{mb}f(x) = e^{2imbx}f(x)$. This means that there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{m,n\in\mathbb{Z}} \big|\langle f, E_{mb}T_{na}g\rangle\big|^2 \leqslant B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}). \tag{1.1}$$

Various characterizations of Gabor frames have been given by Wexler and Raz [15], Daubechise et al. [6], Ron and Shen [14].

It is well known that two Gabor frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are called *dual* of each other if

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \quad \forall f \in L^2(\mathbb{R}). \tag{1.2}$$

Although a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ when $ab < 1$ has infinitely many dual, the standard choice of h is $S^{-1}g$, where $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$Sf = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}g, \quad \forall f \in L^2(\mathbb{R}),$$

is the frame operator of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. The dual generated by $S^{-1}g$ is called the *canonical dual* and other dual are called *alternate duals*. Several duality principles in Gabor frame theory have been proposed; Janssen [12], Gröchenig [10], Casazza et al. [2], Christensen [3], Christensen and Kim [4].

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A well-known characterization of canonical dual Gabor frames is demonstrated in the following proposition.

Proposition 1. *(See [1].) Let $N \in \mathbb{N}$ and let $g \in L^2(\mathbb{R})$ be a function with support in $[0, N]$. Assume that $b \leqslant \frac{1}{N}$ and that there exist $A, B > 0$ such that*

$$A \leqslant G(x) := \sum_{n\in\mathbb{Z}} \left|g(x-na)\right|^2 \leqslant B \quad a.e.\ x.$$

Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, and the canonical dual generator is given by

$$S^{-1}g = \frac{b}{G}g.$$

Due to work by Janssen [13] two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$ if and only if

$$\sum_{k\in\mathbb{Z}} \overline{g(x-n/b-ka)}h(x-ka) = b\delta_{n,0} \quad \text{a.e. } x \in [0, a].$$

Finally, for frames generated by any compactly supported function g whose integer-translates form a partition of unity, e.g., a B-spline, Christensen and Kim constructed a class of dual frame generators, formed by linear combinations of translates of g [3,4]:

Theorem 1.1. *Let $N \in \mathbb{N}$ and $b \in (0, \frac{1}{2N-1}]$. Let $g \in L^2(\mathbb{R})$ be a real-valued bounded function with $\operatorname{supp} g \subseteq [0, N]$, for which*

$$\sum_{n\in\mathbb{Z}} g(x-n) = 1.$$

Then the functions h and k defined by

$$h(x) = bg(x) + 2b\sum_{n=1}^{N-1} g(x+n), \qquad k(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n),$$

where

$$a_0 = b \qquad a_n + a_{-n} = 2b, \quad n = 1, 2, \ldots, N-1,$$

generate two dual frames $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n k\}_{m,n\in\mathbb{Z}}$ for $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$.

In this paper we introduce a general method of constructing alternate duals for a given frame. Our explicit construction can be easily applied for Gabor frames. We show that by choosing an appropriate dual Gabor frame generator, (1.2) provides more precise results.

2. Alternates dual frames

A sequence $\{g_i\}_{i=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there are constants $A, B > 0$ satisfying,

$$A\|f\|^2 \leqslant \sum_{i=1}^{\infty} \left|\langle f, g_i\rangle\right|^2 \leqslant B\|f\|^2, \ (f \in \mathcal{H}). \tag{2.1}$$

If the right-hand side of (2.1) holds, it is said to be a Bessel sequence. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *dual frame* for $\{g_i\}_{i=1}^{\infty}$ if

$$f = \sum_{i=1}^{\infty} \langle f, f_i\rangle g_i. \tag{2.2}$$

The classical choice for $\{f_i\}_{i=1}^{\infty}$ is $\{S^{-1}g_i\}_{i=1}^{\infty}$, where the bounded and invertible frame operator $S : \mathcal{H} \to \mathcal{H}$ is defined by

$$Sf = \sum_{i=1}^{\infty} \langle f, g_i\rangle g_i.$$