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Spatiospectral concentration of vector fields on a sphere

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ABSTRACT

We construct spherical vector bases that are bandlimited and spatially concentrated, or, alternatively, spacelimited and spectrally concentrated, suitable for the analysis and representation of real-valued vector fields on the surface of the unit sphere, as arises in the natural and biomedical sciences, and engineering. Building on the original approach of Slepian, Landau, and Pollak we concentrate the energy of our function bases into arbitrarily shaped regions of interest on the sphere, and within certain bandlimits in the vector spherical-harmonic domain. As with the concentration problem for scalar functions on the sphere, which has been treated in detail elsewhere, a Slepian vector basis can be constructed by solving a finite-dimensional algebraic eigenvalue problem. The eigenvalue problem decouples into separate problems for the radial and tangential components. For regions with advanced symmetry such as polar caps, the spectral concentration kernel matrix is very easily calculated and block-diagonal, lending itself to efficient diagonalization. The number of spatiospectrally well-concentrated vector fields is well estimated by a Shannon number that only depends on the area of the target region and the maximal spherical-harmonic degree or bandwidth. The spherical Slepian vector basis is doubly orthogonal, both over the entire sphere and over the geographic target region. Like its scalar counterparts it should be a powerful tool in the inversion, approximation and extension of bandlimited fields on the sphere: vector fields such as gravity and magnetism in the earth and planetary sciences, or electromagnetic fields in optics, antenna theory and medical imaging.

1. Introduction

Since it is impossible to simultaneously bandlimit and spacelimit a function to a chosen region of interest, we need to resort to functions that are bandlimited but optimally concentrated, with respect to their spatial energy, inside a target region. Slepian, Landau, and Pollak presented the solution to the problem of optimally concentrating a signal in time and frequency in their seminal papers [1–4]. Their construction leads to a family of orthogonal taper functions that have been widely applied as windows to regularize the quadratic inverse problem of power-spectral estimation from time-series observations of finite extent [5]. The "Slepian functions", as we shall be calling them, are furthermore of great utility as a basis for function representation, approximation, interpolation and extension, and to solve stochastic linear inverse problems in a wide range of disciplines. Several authors have studied the time–scale and time–frequency concentration problem in more general settings (see [6–8] and references therein for a review). More specifically, spherical scalar Slepian functions, spatially concentrated while bandlimited, or spectrally concentrated while spacelimited, have been applied in physical, computational, and biomedical fields such as geodesy [9–12] and gravimetry [13–17], geomagnetism [18,19] and geodynamics [20],

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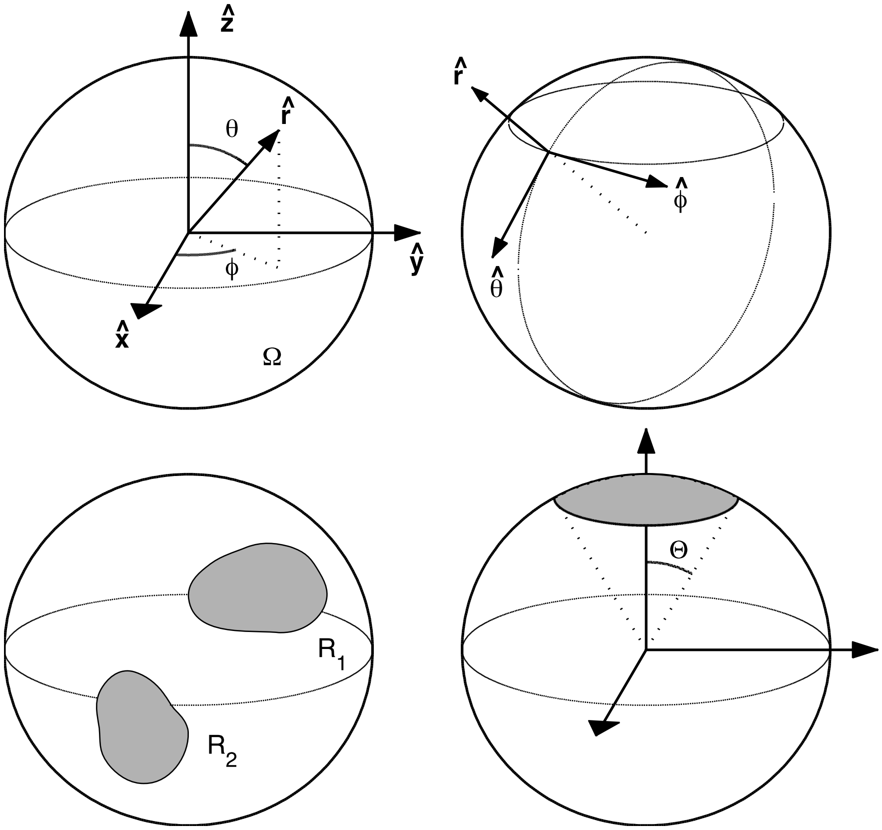

Fig. 1. Sketch illustrating the geometry of the vector spherical concentration problem. Lower right shows an axisymmetric polar cap of colatitudinal radius Θ as treated in Section 4. The area of the region of concentration, $R = R_1 \cup R_2 \cup \cdots$, is denoted by A in the text.

planetary [21–24] and biomedical science [25,26], cosmology [27,28], and computer science [29,30], while continuing to be of interest in information and communication theory [31], signal processing [32,33], and mathematics [34,35].

To date only a few attempts have been made to bring the advantages of spherical Slepian functions into the realm of spherical vector fields. The first successful construction of spatially concentrated bandlimited tangential spherical vector fields was reported for applications in magnetoencephalography [25,26,36]. In geodesy, Eshagh [37] has developed methods to explicitly evaluate the product integrals arising in the concentration problem whose solutions are the vectorial Slepian functions. In this paper we present a complete extension of Slepian's spatiospectral concentration problem to vector fields on the sphere, and give suggestions and examples as to their usage for problems of a geomagnetic nature (e.g. [38,39]). The family of optimally concentrated spherical vectorial multitapers that we will construct in the following should be useful in many scientific applications. In particular in geomagnetism, one of the objectives of the Swarm mission [40] is to model the lithospheric magnetic field with maximal resolution and accuracy, even in the presence of contaminating signals from secondary sources. In addition, and more generally, lithospheric-field data analysis will have to successfully merge information from the global to the regional scale. In the past decade or so, a variety of global-to-regional modeling techniques have come of age, including harmonic splines [41–43], stitching together local models [44–48], and wavelets [49–51]. Due to their optimal combination of spatial locality and spectral bandlimitation the basis functions constructed in this paper should be well suited to combine global and local data while respecting their bandlimitation.

2. Preliminaries

Fig. 1 shows the geometry of the unit sphere $\Omega = \{\hat{\boldsymbol{r}}\colon\ \|\hat{\boldsymbol{r}}\| = 1\}$ and its tangential vectors. The colatitude of spherical points $\hat{\boldsymbol{r}}$ is denoted by $0 \leqslant \theta \leqslant \pi$ and the longitude by $0 \leqslant \phi < 2\pi$; we denote the unit vector pointing outwards in the radial direction by $\hat{\boldsymbol{r}}$, and the unit vectors in the tangential directions towards the south pole and towards the east will be denoted by $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$, respectively. The symbol R will be used to denote a region of the unit sphere Ω, of area $A = \int_R d\Omega$, within which the bandlimited vector field shall be concentrated. The region can be a combination of disjoint subregions, $R = R_1 \cup R_2 \cup \cdots$, and the boundaries of those subregions can be irregularly shaped, as depicted. We will denote the region complementary to R by $\Omega \setminus R$.