



Geometric separation by single-pass alternating thresholding



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ABSTRACT

Modern data is customarily of multimodal nature, and analysis tasks typically require separation into the single components. Although a highly ill-posed problem, the morphological difference of these components sometimes allow a very precise separation such as, for instance, in neurobiological imaging a separation into spines (pointlike structures) and dendrites (curvilinear structures). Recently, applied harmonic analysis introduced powerful methodologies to achieve this task, exploiting specifically designed representation systems in which the components are sparsely representable, combined with either performing ℓ_1 minimization or thresholding on the combined dictionary.

In this paper we provide a thorough theoretical study of the separation of a distributional model situation of point- and curvilinear singularities exploiting a surprisingly simple single-pass alternating thresholding method applied to the two complementary frames: wavelets and curvelets. Utilizing the fact that the coefficients are *clustered geometrically*, thereby exhibiting *clustered/geometric sparsity* in the chosen frames, we prove that at sufficiently fine scales arbitrarily precise separation is possible. Even more surprising, it turns out that the thresholding index sets converge to the wavefront sets of the point- and curvilinear singularities in phase space and that those wavefront sets are perfectly separated by the thresholding procedure. Main ingredients of our analysis are the novel notion of *cluster coherence* and *clustered/geometric sparsity* as well as a *microlocal analysis viewpoint*.

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1. Introduction

Along with the deluge of data we face today, it is not surprising that the complexity of such data is also increasing. One instance of this phenomenon is the occurrence of multiple components, and hence, analyzing such data typically involves a separation step. One most intriguing example comes from neurobiological imaging, where images of neurons from Alzheimer infected brains are studied with the hope to detect specific artifacts of this disease. The prominent parts of images of neurons are spines (pointlike structures) and dendrites (curvelike structures), which require separate analyzes, for instance, counting the number of spines of a particular shape, and determining the thickness of dendrites [32,35].

From an educated viewpoint, it seems almost impossible to extract two images out of one image; the only possible attack point being the morphological difference of the components. The new paradigm of sparsity, which has lately led to some spectacular successes in solving such underdetermined systems, does provide a powerful means to explore this

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difference. The main sparsity-based approach towards solving such separation problems consists in carefully selecting two representation systems, each one providing a sparse representation of one of the components and both being incoherent with respect to the other – the encoding of the morphological difference –, followed by a procedure which generates a sparse expansion in the dictionary combining the two representation systems. This intuitively automatically forces the different components into the coefficients of the ‘correct’ representation system.

Browsing through the literature, the two main sparsity-based separation procedures can be identified to be ℓ_1 minimization (see, e.g., [2,15–17,19–23,28,37–39,41]) and thresholding (see, e.g., [1,21,33,34]). For general papers on ℓ_1 minimization techniques we refer to [7,9,14,13,12] and thresholding to [40] or the reference list in the beautiful survey paper [3]. While ℓ_1 minimization has produced very strong theoretical results, thresholding is typically significantly harder to analyze due to its iterative nature. However, thresholding algorithms are in general much faster than ℓ_1 minimization, which makes them particularly attractive for the aforementioned neurobiological imaging application due to its large problem size.

In this paper we focus on thresholding as a separation technique for separating point- from curvelike structures using radial wavelets and curvelets; in fact, we study the very simple technique of single-pass alternating thresholding, which expands the image in wavelets, thresholds and reconstructs the point part, then expands the residual in curvelets, thresholds and reconstructs the curve part. In this paper we aim for a fundamental mathematical understanding of the precision of separation allowed by this thresholding method. Interestingly, our analysis requires the notions of *cluster coherence* and *clustered/geometrical sparsity*, which were introduced in [18] by the author and Donoho in the context of analyzing ℓ_1 minimization as a separation methodology.

We find the results in our paper quite surprising in two ways. First, the thresholding procedure we consider is very simple, and researchers on thresholding algorithms might at first sight dismiss such single-pass alternating thresholding methodology. Therefore, it is intriguing to us, that we derive a quite similar perfect separation result (Theorem 1.1) as in our paper [18], where ℓ_1 minimization as a separation technique was analyzed. Secondly, to our mind, it is even more surprising that in Theorems 1.2 and 1.3 we derive even more satisfying results by showing that the thresholding index sets converge to the wavefront sets of the point- and curvilinear singularities in phase space and that those wavefront sets are perfectly separated by the thresholding procedure. This, we already suspected for ℓ_1 minimization to be true. However, we are not aware of any analysis tools strong enough to derive these results for separation by ℓ_1 minimization.

1.1. A Geometric Separation Problem

Let us start by defining the following simple but clear model problem of geometric separation (compare also the problem posted in [18]). Consider a ‘pointlike’ object $\mathcal{P}$ made of point singularities:

$$\mathcal{P} = \sum_{i=1}^P |x - x_i|^{-3/2}. \quad (1.1)$$

This object is smooth away from the P given points $(x_i: 1 \leq i \leq P)$. Consider as well a ‘curvelike’ object $\mathcal{C}$, a singularity along a closed curve $\tau: [0, 1] \mapsto \mathbf{R}^2$:

$$\mathcal{C} = \int \delta_{\tau(t)}(\cdot) dt, \quad (1.2)$$

where δ_x is the usual Dirac delta function located at x . The singularities underlying these two distributions are geometrically quite different, but the exponent $3/2$ is chosen so the energy distribution across scales is similar; if $\mathcal{A}_r$ denotes the annular region $r < |\xi| < 2r$,

$$\int_{\mathcal{A}_r} |\hat{\mathcal{P}}|^2(\xi) \asymp r, \quad \int_{\mathcal{A}_r} |\hat{\mathcal{C}}|^2(\xi) \asymp r, \quad r \rightarrow \infty.$$

This choice makes the components comparable as we go to finer scales; the ratio of energies is more or less independent of scale. Separation is challenging at every scale.

Now assume that we observe the ‘Signal’

$$f = \mathcal{P} + \mathcal{C},$$

however, the component distributions $\mathcal{P}$ and $\mathcal{C}$ are unknown to us.

Definition 1.1. The *Geometric Separation Problem* requires to recover $\mathcal{P}$ and $\mathcal{C}$ from knowledge only of f ; here $\mathcal{P}$ and $\mathcal{C}$ are unknown to us, but obey (1.1), (1.2) and certain regularity conditions on the curve τ .

As there are two unknowns ($\mathcal{P}$ and $\mathcal{C}$) and only one observation (f), the problem seems improperly posed. We develop a principled, rational approach which provably solves the problem according to clearly stated standards.

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