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Affine dual frames and Extension Principles

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ABSTRACT

In this work we provide three new characterizations of affine dual frames constructed from refinable functions. The first one is similar to Daubechies et al. (2003) [10, Proposition 5.2] but without any decay assumptions on the generators of a pair of affine systems. The second one reveals the geometric significance of the Mixed Fundamental function and the third one shows that the Mixed Oblique Extension Principle actually characterizes dual framelets. We also extend recent results on the characterization of affine Parseval frames obtained in Stavropoulos (2012) [27, Theorem 2.3].

1. Introduction

Extension Principles were first proposed by Ron and Shen [25,26] and were subsequently extended by Daubechies et al. [10] in the form of the Oblique Extension Principle. OEP relaxes the requirements for the construction of wavelets arising from a pair of refinable functions or from a single refinable function extending thus Mallat's construction of wavelets from orthonormal scaling functions. Extension Principles are important because they can be used to construct wavelets from refinable functions which may not be scaling functions (in the sense that their integer translates may not form a frame but only a Bessel system) with desirable properties such as symmetry and antisymmetry, smoothness or compact support.

In this paper we study the geometric structure associated with bi-framelets arising from pairs of refinable functions. We also show how the Mixed Fundamental function arises from a weak form of a reduction of redundancy between the spaces generated by the integer translates of the pair of refinable functions and those spanned by the detail spaces of scales $j \geqslant 0$. More details about the significance of Extension Principles can be found at [3,5,10,25,26]. We also mention the earliest pioneering works [15,11] on the construction of affine dual frames using Oblique Extension Principle. Apart from Extension Principles various design strategies have been developed for constructing multiscale representations with desirable properties such as good spatial localization, high regularity, arbitrary smoothness, see [1,2,18–22,28,23] and references therein. We end this brief discussion of the Extension Principles literature and of related constructions with the pioneering ϕ-transform of Frazier et al. [13] generalized in the form of dual families of pseudoframes of translates [20].

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We begin with some necessary notation. Let $L_2 := L_2(\mathbb{R}^s)$ be the Hilbert space of all measurable square integrable functions on $\mathbb{R}^s$ with usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_2$. We define the Fourier transform of an integrable function $f : \mathbb{R}^s \to \mathbb{C}$ by

$$\widehat{f}(\gamma) = \int_{\mathbb{R}^s} f(x) e^{-2\pi i x\cdot\gamma}\, dx \quad \bigl(\gamma \in \mathbb{R}^s\bigr),$$

where $x \cdot \gamma$ is the usual inner product on $\mathbb{R}^s$ and we extend the Fourier transform on L_2 as usual. We say that a matrix A of order $s \times s$ is *expansive* if it has integer entries and the eigenvalues of A are bigger than one in modulus. By A^* we denote the Hermitian transpose of A. We define the dilation operator on L_2 with respect to an expansive matrix A by $D_A f = |\det A|^{1/2} f(A\cdot)$. The shift operator on L_2 is defined by $\tau_k f = f(\cdot - k)$, $k \in \mathbb{Z}^s$.

Throughout this paper we assume that ϕ and ϕ^d are two functions in L_2 with the following properties:

(i) the functions $\widehat{\phi}$ and $\widehat{\phi}^d$ are continuous in a neighborhood of the origin and $\lim_{\gamma\to 0} \widehat{\phi}(\gamma) = \lim_{\gamma\to 0} \widehat{\phi}^d(\gamma) = 1$;
(ii) the $\mathbb{Z}^s$-periodic functions $\Phi = \sum_{k\in\mathbb{Z}^s} |\widehat{\phi}(\cdot + k)|^2$ and $\Phi^d = \sum_{k\in\mathbb{Z}^s} |\widehat{\phi}^d(\cdot + k)|^2$ belong in $L_\infty(\mathbb{T}^s)$, the space of all measurable essentially bounded functions on $\mathbb{T}^s = [0,1)^s$; and
(iii) the functions ϕ and ϕ^d are *refinable* with respect to an expansive matrix A, i.e. there exist two $\mathbb{Z}^s$-periodic functions H_0 and H_0^d in $L_2(\mathbb{T}^s)$ (the space of all square integrable functions on $\mathbb{T}^s$) called *low pass filters* or *refinement masks* such that

$$\widehat{\phi}\bigl(A^*\gamma\bigr) = H_0(\gamma)\widehat{\phi}(\gamma) \quad \text{and} \quad \widehat{\phi}^d\bigl(A^*\gamma\bigr) = H_0^d(\gamma)\widehat{\phi}^d(\gamma)$$

up to a null set with respect to the Lebesgue measure on $\mathbb{R}^s$. For the above definition of Φ and Φ^d we denote the *spectrum* of ϕ and ϕ^d by

$$\sigma(\phi) = \bigl\{\gamma \in \mathbb{T}^s\colon\ \Phi(\gamma) \neq 0\bigr\} \quad \text{and} \quad \sigma\bigl(\phi^d\bigr) = \bigl\{\gamma \in \mathbb{T}^s\colon\ \Phi^d(\gamma) \neq 0\bigr\}$$

and we denote by V_0 and V_0^d the closed linear span of the sets $\{\phi(\cdot - n)\colon\ n \in \mathbb{Z}^s\}$ and $\{\phi^d(\cdot - n)\colon\ n \in \mathbb{Z}^s\}$ respectively.

We also consider two finite sets of refinable functions in L_2 whose elements are called *wavelets*, namely $\Psi = \{\psi_i\colon\ i = 1, \dots, m\}$ and $\Psi^d = \{\psi_i^d\colon\ i = 1, \dots, m\}$ such that

$$\widehat{\psi}_i\bigl(A^*\gamma\bigr) = H_i(\gamma)\widehat{\phi}(\gamma) \quad \text{and} \quad \widehat{\psi_i^d}\bigl(A^*\gamma\bigr) = H_i^d(\gamma)\widehat{\phi}^d(\gamma) \quad \text{a.e. } \gamma \in \mathbb{R}^s,$$

where H_i and H_i^d are $\mathbb{Z}^s$-periodic functions in $L_2(\mathbb{T}^s)$ called *high pass filters* or *wavelet masks*. For the above selection of wavelets $\psi \in \Psi$ and $\psi^d \in \Psi^d$ we denote the set

$$X_\Psi = \bigl\{\psi_{i,j,k} = D_A^j \tau_k \psi_i\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^s,\ i = 1, \dots, m\bigr\}.$$

The corresponding notation for the set X_{Ψ^d} is similar. The set X_Ψ is called a *homogeneous wavelet family* or *affine family* generated from the *mother wavelets* $\psi \in \Psi$. If there exist two positive constants c and C such that for any $f \in L_2$ we have

$$c\|f\|_2^2 \leqslant \sum_{\psi\in X_\Psi} \bigl|\langle f, \psi\rangle\bigr|^2 \leqslant C\|f\|_2^2,$$

then we say that X_Ψ is an *affine frame* or *a homogeneous wavelet frame* for L_2 and the elements of X_Ψ are called *framelets*. If $c = C$ then X_Ψ is a *tight frame* and if $c = C = 1$ then X_Ψ is a *Parseval frame*. On the other hand, if only the right-hand side of the above double inequality holds then we say that X_Ψ is a *Bessel system*. If both X_Ψ and X_{Ψ^d} are Bessel systems and for any $f \in L_2$ we have the reconstruction formula

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^s} \sum_{i=1}^{m} \bigl\langle f, \psi_{i,j,k}^d\bigr\rangle \psi_{i,j,k}$$

in the L_2-sense, then we say that X_{Ψ^d} is an *affine dual frame* of X_Ψ (and vice versa) or we simply say that (X_Ψ, X_{Ψ^d}) is a pair of *dual framelets*. We note that in the study of affine dual frames the Bessel property of a wavelet family is important [15, Theorem 2.3]. We also remark that the previous equation implies that each one of the two wavelet families is a frame for L_2 [26, Proposition 1]. If X_Ψ is a Riesz basis of L_2 then the unique dual Riesz basis of X_Ψ may not be a wavelet family [6,8,9]. Therefore the construction of an affine wavelet family which is dual to another affine wavelet family is not automatic.

On the other hand, let φ, φ^d be two functions in L_2 (not necessarily equal to ϕ and ϕ^d) and Ψ, Ψ^d be two sets of wavelet families as above. For any $j_0 \in \mathbb{Z}$ we denote the set:

$$X_{\varphi,\Psi}^{(j_0)} = \bigl\{D_A^j \tau_k \psi\colon\ j \geqslant j_0,\ k \in \mathbb{Z}^s,\ \psi \in \Psi\bigr\} \cup \bigl\{D_A^{j_0} \tau_k \varphi\colon\ k \in \mathbb{Z}^s\bigr\}. \tag{1.1}$$