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On the evaluation of prolate spheroidal wave functions and associated quadrature rules

Andrei Osipov [*,1], Vladimir Rokhlin [2,3]

Departments of Mathematics and Computer Science, Yale University, 51 Prospect st., New Haven, CT 06511, USA

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ABSTRACT

As demonstrated by Slepian et al. in a sequence of classical papers (see Slepian (1983) [33], Slepian and Pollak (1961) [34], Landau and Pollak (1961) [18], Slepian and Pollak (1964) [35], Slepian (1965) [36]), prolate spheroidal wave functions (PSWFs) provide a natural and efficient tool for computing with bandlimited functions defined on an interval. Recently, PSWFs have been becoming increasingly popular in various areas in which such functions occur – this includes physics (e.g. wave phenomena, fluid dynamics), engineering (signal processing, filter design), etc.

To use PSWFs as a computational tool, one needs fast and accurate numerical algorithms for the evaluation of PSWFs and related quantities, as well as for the construction of corresponding quadrature rules, interpolation formulas, etc. During the last 15 years, substantial progress has been made in the design of such algorithms – see, for example, Xiao et al. (2001) [40] (see also Bowkamp (1947) [6], Slepian and Pollak (1961) [34], Landau and Pollak (1961) [18], Slepian and Pollak (1964) [35] for some classical results).

The complexity of many of the existing algorithms, however, is at least quadratic in the band limit c. For example, the evaluation of the nth eigenvalue of the prolate integral operator requires $O(c^2+n^2)$ operations (see e.g. Xiao et al. (2001) [40]); the construction of accurate quadrature rules for the integration (and associated interpolation) of bandlimited functions with band limit c requires $O(c^3)$ operations (see e.g. Cheng et al. (1999) [8]). Therefore, while the existing algorithms are satisfactory for moderate values of c (e.g. $c \leqslant 10^3$), they tend to be relatively slow when c is large (e.g. $c \geqslant 10^4$).

In this paper, we describe several numerical algorithms for the evaluation of PSWFs and related quantities, and design a class of PSWF-based quadratures for the integration of bandlimited functions. While the analysis is somewhat involved and will be published separately (currently, it can be found in Osipov and Rokhlin (2012) [27]), the resulting numerical algorithms are quite simple and efficient in practice. For example, the evaluation of the nth eigenvalue of the prolate integral operator requires $O(n+c\cdot\log c)$ operations; the construction of accurate quadrature rules for the integration (and associated interpolation) of bandlimited functions with band limit c requires $O(c)$ operations. All algorithms described in this paper produce results essentially to machine precision. Our results are illustrated via several numerical experiments.

* Corresponding author.
E-mail address: andrei.osipov@yale.edu (A. Osipov).

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1. Introduction

The principal purpose of this paper is to describe several numerical algorithms associated with bandlimited functions. While these algorithms are quite simple and efficient in practice, the analysis is somewhat involved, and will be published separately (currently the proofs and additional details can be found in [27,28]).

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be bandlimited with band limit $c > 0$ if there exists a function $\sigma \in L^2[-1, 1]$ such that

$$f(x) = \int_{-1}^{1} \sigma(t) e^{icxt}\, dt. \tag{1}$$

In other words, the Fourier transform of a bandlimited function is compactly supported. While (1) defines f for all real x, one is often interested in bandlimited functions whose argument is confined to an interval, e.g. $-1 \leqslant x \leqslant 1$. Such functions are encountered in physics (wave phenomena, fluid dynamics), engineering (signal processing), etc. (see e.g. [33,11,29]).

About 50 years ago it was observed that the eigenfunctions of the integral operator $F_c : L^2[-1, 1] \to L^2[-1, 1]$, defined via the formula

$$F_c[\varphi](x) = \int_{-1}^{1} \varphi(t) e^{icxt}\, dt, \tag{2}$$

provide a natural tool for dealing with bandlimited functions defined on the interval $[-1, 1]$. Moreover, it was observed (see [34,18,35]) that the eigenfunctions of F_c are precisely the prolate spheroidal wave functions (PSWFs), well known from the mathematical physics (see, for example, [24,11]).

Obviously, to use PSWFs as a computational tool, one needs fast and accurate numerical algorithms for the evaluation of PSWFs and related quantities, as well as for the construction of quadratures, interpolation formulas, etc. For the last 15 years, substantial progress has been made in the design of such algorithms – see, for example, [37,40] (see also [6,34,18,35] for some classical results, and [30,38] for some related developments).

The complexity of many of the existing algorithms, however, is at least quadratic in the band limit c. For example, the evaluation of the nth eigenvalue of the prolate integral operator requires $O(c^2 + n^2)$ operations (see e.g. [40]); also, the construction of accurate quadrature rules for the integration (and associated interpolation) of bandlimited functions with band limit c requires $O(c^3)$ operations (see e.g. [8]). Therefore, while the existing algorithms are satisfactory for moderate values of c (e.g. $c \leqslant 10^3$), they tend to be relatively slow when c is large (e.g. $c \geqslant 10^4$).

In this paper, we describe several numerical algorithms for the evaluation of PSWFs and related quantities, and design a class of PSWF-based quadratures for the integration of bandlimited functions. While the analysis is somewhat involved and will be published separately (currently, it can be found in [27]), the resulting numerical algorithms are quite simple and efficient in practice. For example, the evaluation of the nth eigenvalue of the prolate integral operator requires $O(n + c \log c)$ operations; also, the construction of accurate quadrature rules for the integration of bandlimited functions with band limit c requires $O(c)$ operations. In addition, the evaluation of the nth PSWF is done in two steps. First, we carry out a certain precomputation, that requires $O(n + c \log c)$ operations. Then, each subsequent evaluation of this PSWF at a point in $[-1, 1]$ requires $O(1)$ operations.

This paper is organized as follows. Section 2 contains a brief overview. Section 3 contains mathematical and numerical preliminaries to be used in the rest of the paper. Section 4 contains the summary of the principal analytical results of the paper. Section 5 contains the description and analysis of the numerical algorithms for the evaluation of the quadrature rules and some related quantities. In Section 6, we report some numerical results. In Section 7, we illustrate the analysis via several numerical experiments.

2. Overview

In this section, we provide an overview of the paper. More specifically, Section 2.1 is dedicated to the numerical evaluation of PSWFs and related quantities. In Section 2.2, we discuss several existing quadrature rules for the integration of bandlimited functions. In Section 2.3, we introduce a new class of PSWFs-based quadrature rules and describe the underlying ideas. In Section 2.4, we outline the analysis (further details can be found in [27]).

2.1. Numerical evaluation of PSWFs

For any real $c > 0$ and integer $n \geqslant 0$, the corresponding PSWF ψ_n can be expanded into an infinite series of Legendre polynomials (see Section 3.2). The coefficients of such expansions decay superalgebraically (see e.g [40]); in particular, relatively few terms of the Legendre series are required to evaluate $\psi_n(x)$ to essentially the machine precision, for any $-1 \leqslant x \leqslant 1$. The use of this observation for the numerical evaluation of PSWFs goes back at least to the classical Bouwkamp algorithm [6] (see also Section 3.2, in particular Theorem 10 and Remark 9, and [40] for more details).