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Band-limited scaling and wavelet expansions ☆

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Operators $Q_jf=\sum_{n\in\mathbb{Z}}\langle f,\widetilde{\varphi}_{jn}\rangle\varphi_{jn}$ are studied for a class of band-limited functions φ and a wide class of tempered distributions $\widetilde{\varphi}$. Convergence of Q_jf to f as $j\to+\infty$ in the L_2-norm is proved under a very mild assumption on φ, $\widetilde{\varphi}$, and the rate of convergence is equal to the order of Strang–Fix condition for φ. To study convergence in L_p, $p>1$, we assume that there exists $\delta\in(0,1/2)$ such that $\overline{\widehat{\widetilde{\varphi}}}\widehat{\varphi}=1$ a.e. on $[-\delta,\delta]$, $\widehat{\varphi}=0$ a.e. on $[l-\delta,l+\delta]$ for all $l\in\mathbb{Z}\setminus\{0\}$. For appropriate band-limited or compactly supported functions $\widetilde{\varphi}$, the estimate $\|f-Q_jf\|_p\leqslant C\omega_r(f,2^{-j})_{L_p}$, where ω_r denotes the r-th modulus of continuity, is obtained for arbitrary $r\in\mathbb{N}$. For tempered distributions $\widetilde{\varphi}$, we proved that Q_jf tends to f in the L_p-norm, $p\geqslant 2$, with an arbitrary large approximation order. In particular, for some class of differential operators L, we consider $\widetilde{\varphi}$ such that $Q_jf=\sum_{n\in\mathbb{Z}}Lf(2^{-j}\cdot)(n)\varphi_{jn}$. The corresponding wavelet frame-type expansions are found.

1. Introduction

The well-known sampling theorem (which is often called Kotel'nikov's or Shannon's theorem; [30,21,25,34] and even [13]) states that

$$f(x)=\sum_{n\in\mathbb{Z}}f\left(2^{-j}n\right)\frac{\sin\pi(2^jx-n)}{\pi(2^jx-n)} \tag{1}$$

for any function $f\in L_2(\mathbb{R})$ whose Fourier transform is supported on $[-2^{j-1},2^{j-1}]$. This formula is very useful for engineers. It was just Kotel'nikov and Shannon who started to apply the formula for signal processing, respectively in 1933 and 1949. Up to now, an overwhelming diversity of digital signal processing applications and devices are based on it and more than successfully use it. Without sampling theorem it would be impossible to make use of Internet, make photos and videos. On the other hand, (1) is an important and interesting formula for mathematicians. Recently Butzer with co-authors published several papers [4–8], where they analyze sampling theorem and its applications and development. In particular, the equivalence of sampling theorem to some other classical formulas was established for some classes of band-limited functions. Also in [9,10] they studied a generalization of sampling decomposition replacing sinc-function by certain linear combinations of B-splines. Linear summation methods of sampling expansion was studied by Kivinukk and Tamberg in [11,12].

From the point of view of wavelet theory, (2) is not a theorem, it is just an illustration for the Shannon MRA. Indeed, the function $\varphi(x)=\frac{\sin\pi x}{\pi x}$ is a scaling function for this MRA, and a function f belongs to the sample space V_j if and only if

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its Fourier transform is supported on $[-2^{j-1}, 2^{j-1}]$. So, such a function f can be expanded as $f = \sum_{n\in\mathbb{Z}}\langle f, \varphi_{jn}\rangle\varphi_{jn}$, which coincides with (1). Also, since $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA, any $f \in L_2(\mathbb{R})$ can be represented as

$$f = \lim_{j\to+\infty}\sum_{n\in\mathbb{Z}}\langle f, \varphi_{jn}\rangle\varphi_{jn}. \tag{2}$$

Moreover, (2) has an arbitrary large approximation order. This happens because the function $\varphi(x) = \frac{\sin\pi x}{\pi x}$ is band-limited, a similar property cannot be valid for other natural classes of φ, in particular, for compactly supported φ. Some generalizations of this fact will be proved in the present paper.

We are going to consider band-limited functions φ with continuous or discontinuous Fourier transform and study the corresponding scaling expansions (or quasi-projection operators) $\sum_{n\in\mathbb{Z}}\langle f, \widetilde{\varphi}_{jn}\rangle\varphi_{jn}$, where $\widetilde{\varphi}$ is, generally speaking, a tempered distribution, in particular, $\widetilde{\varphi}$ may be from the same class of band-limited functions.

The operators $Q_j f = \sum_{n\in\mathbb{Z}}\langle f, \widetilde{\varphi}_{jn}\rangle\varphi_{jn}$ appear very often in the papers concerned with wavelets. Probably one of the first appearing was in the well-known paper [14] by Cohen, Daubechies and Feauveau, where a method for the construction of biorthogonal wavelet bases was developed. In this case the functions φ, $\widetilde{\varphi}$ are in L_2, refinable, and their integer translations are biorthogonal. A method for the construction dual wavelet frames was developed in [27,28] by Ron and Shen, where these operators also play an important role. In this case the integer translations of φ, $\widetilde{\varphi}$ should not be biorthogonal, but the function are still in L_2 and refinable. Convergence and approximation properties of Q_j, with compactly supported $\varphi, \widetilde{\varphi}$ were actively studied by many authors (see [1–3,18,22,19,20,26] and the references therein). Polynomial reproducibility plays a vital role in these results. The most general results for L_p-convergence were obtained by Jia in [20] who proved that

$$\|f - Q_j f\|_p \leqslant C\omega\left(f, 2^{-j}\right) \quad \forall f \in W_p^k,$$

where ω denotes the modulus of continuity, under the assumptions: $\varphi, \widetilde{\varphi}$ are compactly supported, $\varphi \in L_p$, $\widetilde{\varphi} \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, and Q_0 reproduces polynomials of degree $k-1$. The method based on polynomial reproducibility is not appropriate for slowly decaying functions, such as functions φ whose Fourier transform is discontinuous. Another approach was employed by Jetter and Zhou [16,17], and developed in [26], where Fourier transform technique was applied. The results of the present paper are obtained with using the latter method which allows to work with a wide class of band-limited functions φ and with a wide class of tempered distributions $\widetilde{\varphi}$.

The following notations will be used throughout the paper. The Schwartz class of functions defined on $\mathbb{R}$ is denoted by S, and S' is the dual space of S, i.e. the corresponding space of tempered distributions. We shall use the basic notion and facts from distribution theory which can be found, e.g., in [15] or [33]. If $f \in S$, $g \in S'$, then $\overline{\langle f, g\rangle} := \langle g, f\rangle := g(f)$. If $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$, $\frac{1}{p} + \frac{1}{q} = 1$, then $\langle f, g\rangle := \int_{\mathbb{R}^d} f\bar{g}$. If $f \in S'$, then $\widehat{f}$ denotes its Fourier transform defined by $\langle \widehat{f}, \widehat{g}\rangle = \langle f, g\rangle$, $g \in S$. If f is a function defined on $\mathbb{R}$, we set

$$f_{jk}(x) := 2^{j/2} f\left(2^j x + k\right), \quad j \in \mathbb{Z},\ k \in \mathbb{R}.$$

If $f \in S'$, $j \in \mathbb{Z}$, $k \in \mathbb{R}$, we define f_{jk} by

$$\langle f_{jk}, g\rangle = \langle f, g_{-j,-2^{-j}k}\rangle \quad \forall g \in S.$$

For convenience, sometimes we will write $2^{j/2}f(2^j x + k)$ instead of f_{jk} even for $f \in S'$.

If $\langle f, \widetilde{\varphi}_{jk}\rangle$ has meaning and the series $\sum_{k\in\mathbb{Z}}\langle f, \widetilde{\varphi}_{jk}\rangle\varphi_{jk}$ converges in some sense, we set

$$Q_j(\varphi, \widetilde{\varphi}, f) = Q_j(f) := \sum_{k\in\mathbb{Z}}^{d}\langle f, \widetilde{\varphi}_{jk}\rangle\varphi_{jk}.$$

Let $\varphi \in S'$, its Fourier transform $\widehat{\varphi}$ be defined on $\mathbb{R}$ and n-times differentiable on $\mathbb{Z}$. One says that the Strang–Fix condition of order n holds for φ if

$$\frac{d^k\widehat{\varphi}}{d\xi^k}(l) = 0, \quad k = 0, \dots, n,$$

for all $l \in \mathbb{Z}$, $l \neq 0$.

We use W_p^n, $1 \leqslant p < \infty$, $n \in \mathbb{N}$, to denote the Sobolev space on $\mathbb{R}$, i.e. the set of functions whose derivatives up to order n are in $L_p(\mathbb{R})$, with usual Sobolev norm.

We use ∇_t to denote the difference operator given by $\nabla_t f = f(\cdot) - f(\cdot - t)$. The n-th modulus of continuity of a function f in $L_p(\mathbb{R})$ is defined by

$$\omega_n(f, h)_{L_p} = \sup_{|t|\leqslant h}\left\|\nabla_t^n f\right\|_p, \quad h \geqslant 0.$$

If F is a 1-periodic function and F in $L([0, 1])$, then $\widehat{F}(k) = \int_0^1 F(x)e^{-2\pi ikx}\,dx$ is its k-th Fourier coefficient.