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An ultra-fast smoothing algorithm for time–frequency transforms based on Gabor functions

K. Hallatschek

Max-Planck-Institute for Plasma Physics, EURATOM–IPP Association, Boltzmannstr. 2, 85748 Garching, Germany

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ABSTRACT

Gabor functions, Gaussian wave packets, are optimally localized in time and frequency, and thus in principle ideal as (frame) basis functions for a wavelet, windowed Fourier or wavelet-packet transform for the detection of events in noisy signals or for data compression. A major obstacle for their use is that a tailored efficient operator acting on the transform coefficients for altering the width of the wave packets does not exist. However, by virtue of a curious property of the Gabor functions it is possible to change the width of the wave packets using just one-dimensional convolutions with very short kernels. The cost of a wavelet-type transform based on the scheme presented below is similar to that of a low order wavelet transform for a compact kernel and significantly less than the algorithme à trous. The scheme can hence easily be employed for the processing of signals in real time.

1. Introduction

For the detection of events, i.e., conspicuous or *improbable* amplitudes in an otherwise noisy signal it is necessary to restrict the attention to certain classes of events, since after all, every particular noise signal in itself is highly improbable and yet insignificant since the improbability is balanced by the quantity of possible noise wave-forms. Assuming white Gaussian background noise (which can often be achieved by suitable filtering), it is optimal for the purpose of event detection in the spirit of Wiener filtering to form scalar products of the signal with the possible event wave-forms, which vary in width, frequency and position. The set of scalar products with all test-functions can be regarded as a specific time–frequency transform, the coefficients of which are to be scanned subsequently for amplitudes significantly above the noise level. For constant width of the event wave-forms or test functions we arrive at a windowed Fourier transform [4]. Since for physical signals the duration of an event and its frequency are likely caused by similar time scales, it is better to vary the width of the test-functions proportional to their frequency, which yields a wavelet transform [5]. Lacking knowledge about the event durations it may even be best to test varying widths, which results in the framework of the wavelet-packet transform [6].

In all cases but the windowed Fourier transform, it is necessary to compute scalar products of the signal with test functions of increasing time duration or width, which in principle lends itself to the fast wavelet transform and the related algorithms. These algorithms are based on the recursive separation of smooth and fine signal components by discrete low- and high-pass filters. Unfortunately, the frequency resolution of the usual fast wavelet transforms [5] is fixed at one octave, which is generally to coarse for signals such as those observed in plasma physics [7–10]. Additionally, the resolution deteriorates owing to the leakage between far apart octaves due to imperfect filtering before the subsampling step in the wavelet transform. Apart from the frequency resolution, the spatial resolution is suboptimal, because it is above the Heisenberg limit $\Delta\omega\Delta t < 1$.

E-mail address: Klaus.Hallatschek@ipp.mpg.de.

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On the other hand, it is frequently reasonable to assume the event generating process to be stationary over certain times (the duration of the event), such as is typical, e.g., for natural sound, or in our case for certain types of events in magnetic fusion devices. While exact stationarity would imply that the events consist of superpositions of harmonic functions, this is only true in an approximate sense, due to the finite time duration. To construct acceptable wave-forms for events, one is lead to search for wave functions of finite duration which are as close as possible to a harmonic, i.e., which have the smallest possible width in frequency space. In that respect, the Gaussian wave packets are an optimum compromise between finite duration and resemblance to a harmonic function in the sense that they have the minimum product of time and frequency variance.

The corresponding windowed Fourier transform is the Gabor transform [11]. The definition of this transform can be readily adapted to accommodate for a varying width of the test functions as required in the wavelet or wavelet-packet transform. Unfortunately, a Gabor wavelet at given width is not the discrete sum of Gabor wavelets at different width, which is the basis for the recursion of the fast wavelet transform algorithms. One can still apply a similar recursion, the "algorithme à trous" [12], which is based on the approximate representation of increasingly wider wavelets by piecewise polynomials. Somewhat depending on the required accuracy this requires however a rather significant amount of coefficients per wavelet, rendering it much slower than the fast wavelet transform.

A more efficient method is to represent the Gabor functions of a particular width directly as approximate sums of Gabor functions of another width, which in terms of numerical accuracy can in principle be optimally done by employing the theory of frames [4]. The resulting coefficients allow the recursive computation of Gabor coefficients for increasing width (the smoothing operation), starting from the smallest one, which allows the efficient computation of coefficient sets corresponding to a multi-voice wavelet transform [4] or even a wavelet-packet transform [6].

An even faster method proposed in the present paper can however be obtained by capitalizing on an identity for discrete convolutions of Gabor functions. For certain discrete values of the phase space density, this reduces the computational cost for the smoothing operation to just a dozen floating point multiply-adds per real coefficient of the transform at a relative precision of 10^{-4}, which is in the range of the fast wavelet transform for low order wavelets.

The Balian–Low theorem [14] implies that the phase space density of Gabor functions to be used for any numerically sound approach to Gabor-type transforms should be *above* the limiting density of $(\Delta\omega\Delta t)^{-1}_{\text{crit}} = 2\pi$, which is of course also a condition for the algorithm to be discussed below. Interestingly, it turns out that there are discrete values of the phase space density, which are particularly well suitable for a fast Gabor transform.

We start by defining the desired transform coefficients, followed by the start-up operation, which delivers the windowed Fourier transform to undergo subsequent refinement in frequency and coarsening in time. In the following Section 4 the central identity is derived, which relates any discrete convolution of Gabor functions in real space to a discrete convolution of Gabor functions in frequency space. From this follow preferred vertex densities in time and frequency, which allow the two convolutions to be directly represented. With these grids in the time–frequency plane, it is possible to convert the discrete convolutions to exact convolutions (Section 6). The paper concludes by touching the reversion and stability of the scheme.

2. Statement of the problem

As discussed in the introduction, given a discrete signal $s_k \in \mathbb{C}$, $k \in \mathbb{Z}$ we would like to compute the discrete Gabor coefficients

$$c_{t,\omega,\sigma} := \sum_{j\in\mathbb{Z}} s_j g^*_{\sigma,\omega}(j-t), \qquad g_{\sigma,\omega}(t) := g(t/\sigma)e^{i\omega t}, \qquad g(x) := e^{-x^2/2} \tag{1}$$

for certain regular grids of points (k,ω) in the time–frequency plane and sets of widths σ. Due to the fast decay of the Gabor functions, the series effectively can be regarded as a finite sum. (A severe general practical problem of the wavelet transform turned out to be variable sampling rates of the signal sources, which require significantly higher complexity to be satisfactorily transferred to Fourier space [15,16].)

To have a concrete problem in view, let us define a specific set of coefficient indices (Fig. 1) corresponding to a multi-voice wavelet transform (for which also a real time application of the algorithm has been implemented [1–3]),

$$M(\nu, \Delta t_0, \sigma_0) := \big\{(t,\omega,\sigma) = \iota(k,l,\mu) \,\big|\, k \in \mathbb{Z},\ l \in \{0,\dots,\nu-1\},\ \mu \in \mathbb{N}_0\big\}, \tag{2}$$

$$\iota(k,l,\mu) := \left(\left(k+\frac{1}{2}\right)\Delta t_\mu, \left(2\nu - l - \frac{1}{2}\right)\Delta\omega_\mu, \sigma_\mu\right), \tag{3}$$

$$\Delta\omega_\mu = 2^{-\mu}\Delta\omega_0, \qquad \Delta t_\mu = 2^\mu \Delta t_0, \qquad \sigma_\mu = 2^\mu \sigma_0, \tag{4}$$

where k is the time index, l the frequency index within one octave, μ the octave index, and ν the number of voices per octave. (The shifts by 1/2 can be understood looking at Fig. 1 by taking into account that it is preferable to have only one class of vertices within an octave.) $\Delta\omega_0$ and Δt_0 are free parameters and control the density of vertices in time and frequency. Independent of the set of vertices in the time–frequency plane, σ_0 is the with of a Gabor function at octave $l = 0$.