Appl. Comput. Harmon. Anal. 26 (2009) 283–290

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Letter to the Editor

Chirp sensing codes: Deterministic compressed sensing measurements for fast recovery

Lorne Applebaum[a], Stephen D. Howard[b], Stephen Searle[c], Robert Calderbank[a,*]

[a] Electrical Engineering, Princeton University, B318 Engineering Quad, Princeton, NJ 08544, USA
[b] Defence Science & Technology Organisation, PO Box 1500, Edinburgh 5111, Australia
[c] Electronic & Electrical Engineering, The University of Melbourne, Victoria 3010, Australia

ARTICLE INFO

Article history:
Available online 2 September 2008
Communicated by Charles K. Chui on 5 April 2008

Keywords:
Compressed sensing
Deterministic measurement matrices
Chirp detection

ABSTRACT

Compressed sensing is a novel technique to acquire sparse signals with few measurements. Normally, compressed sensing uses random projections as measurements. Here we design deterministic measurements and an algorithm to accomplish signal recovery with computational efficiency. A measurement matrix is designed with chirp sequences forming the columns. Chirps are used since an efficient method using FFTs can recover the parameters of a small superposition. We show that this type of matrix is valid as compressed sensing measurements. This is done by bounding the eigenvalues of sub-matrices, as well as an empirical comparison with random projections. Further, by implementing our algorithm, simulations show successful recovery of signals with sparsity levels similar to those possible by matching pursuit with random measurements. For sufficiently sparse signals, our algorithm recovers the signal with computational complexity $O(K \log K)$ for K measurements. This is a significant improvement over existing algorithms.

1. Introduction

The sparsity of signals is a fact often exploited in signal processing. In particular, the common way to compress a signal is to transform it to the basis in which it is sparse and subsequently store only the locations and values of the few non-zero elements. Recently, it has been discovered that, in addition to storage, signal sparsity can be leveraged to reduce the number of measurements for signal acquisition and detection. It has been shown that, if a signal is sufficiently sparse, a small number projections onto random vectors is enough to recover the signal [2,6]. This method has been called *Compressed Sensing*.

In compressed sensing, the use of randomly generated projections to make measurements has the useful consequence of sidestepping the computationally difficult task of checking whether the measurements allow for signal recovery. By considering recovery stocastically, it has been shown that measurements generated from Gaussian or Bernoulli random variables allow for signal recovery with high probability. In some ways, the use of random measurements may be viewed as an analogy to random codes used by Shannon to prove theorems in channel coding. Though useful in proofs, purely random channel codes are never used in practice because encoding and decoding would be far too computationally intensive. Instead, practical channel codes are developed with an efficient coding and decoding scheme in mind. We have a similar situation in compressed sensing. Though ℓ_1 minimization has been shown to recover the signal from random projections [2], it is computationally expensive. The question arises as to whether we can design projections to facilitate the rapid recovery

* Corresponding author.
E-mail address: calderbk@math.princeton.edu (R. Calderbank).

1063-5203/$ – see front matter
doi:10.1016/j.acha.2008.08.002

of the signal. This is an issue of practical consequence. If compressed sensing is to be used in real-time systems, we must have a method which, in addition to reducing the number of measurements, is able to recover the signal quickly. Here we present a proof of concept scheme which accomplishes this.

A number of decoding schemes have been proposed that improve upon the ℓ_1 minimization signal recovery technique (also known as *basis pursuit*). However, most schemes presuppose random measurements. Examples include orthogonal matching pursuit [13] and its refinements [8]. In contrast, the scheme presented here exploits structure in deterministically designed measurements to make recovery much faster. There exists a small number of other schemes with less structurally random measurements [5,11,12]. The scheme presented here has lower recovery complexity.

The remainder of the paper is organized as follows. In Section 2 we provide necessary background and notation and in Section 3 we introduce our encoding scheme and the corresponding decoding algorithm. Section 4 provides analysis of our encoding matrix in terms of restricted isometry properties commonly employed in compressed sensing. In Section 5 we consider our scheme in the special case of Fourier signals and present a modification to improve the scheme's robustness. In Sections 6 and 7 we examine our algorithm in terms of computational complexity, signal recovery and robustness to noise.

2. Compressed sensing background and notation

We consider discrete signals of finite length. Let x be a length N signal which we would like to sense and recover. We assume that x is sparse in some orthonormal basis. Thus, we can write x as

$$x = \Psi s \tag{1}$$

where s is a length N vector with fewer than M non-zero elements. We measure x with $K < N$ projections which have results given in the vector y. The vectors projected upon are set as the rows of the $K \times N$ matrix Φ which gives

$$y = \Phi\Psi s = \Theta s \tag{2}$$

where the second equality is by definition of Θ. We are free to design Φ and thus Θ. Though, if we design Θ we should remain aware that actual sensing of the signal is done with Φ.

Since Θ is a wide matrix, solving for s given y is ill posed. However, using non-linear methods, we can leverage the fact that s has at most M non-zero elements. It has been shown in [3] that if Θ satisfies certain *restricted isometry properties* (RIP), s can be recovered perfectly using an ℓ_1 minimization. An important example is randomly generated Θ. Several results exist showing that, when M satisfies

$$M < cK/\log(N/K), \tag{3}$$

with a known constant c, randomly generated matrices of various types satisfy RIP with high probability [1]. Thus, if a signal's sparsity is bounded by (3), then it can be recovered from K random measurements with high probability.

We will consider our designed Θ more precisely in terms of RIP in Section 4. There, we will also give an empirical comparison of the eigenvalue statistics of our designed Θ with those of randomly generated measurements showing that (3) applies to recoverability from chirp sensing codes.

3. Chirped sensing codes

We approach the recovery problem by noting that finding s is equivalent to discovering which small linear combinations of the columns of Θ form y. We will design Θ to facilitate this. In particular, we will look at a Θ designed with chirp signals forming the columns.

A length K chirp signal has the form

$$v_{m,r}(l) = \alpha \cdot e^{\frac{j2\pi ml}{K} + \frac{j2\pi rl^2}{K}}, \quad m, r \in \mathbb{Z}_K, \tag{4}$$

where m is the base frequency and r is the chirp rate. For a length K signal, there are K^2 possible pairs (m, r). We will form a $K \times K^2$ sized Θ which has columns filled with all K^2 uni-modular chirp signals (setting $\alpha = 1$ for notational convenience, though in Section 4 $\alpha = \frac{1}{\sqrt{K}}$ is used).

Consider a vector y, indexed by l, formed from the linear combination of some chirp signals

$$y(l) = s_1 e^{\frac{j2\pi m_1 l}{K} + \frac{j2\pi r_1 l^2}{K}} + s_2 e^{\frac{j2\pi m_2 l}{K} + \frac{j2\pi r_2 l^2}{K}} + \cdots \tag{5}$$

which have base frequencies defined by m_i and chirp rates defined by r_i. The chirp rates can be recovered from y by looking at $\bar{y}(l)y(l+T)$, where the index $l+T$ is taken mod K. This gives

$$f(l) = \bar{y}(l)y(l+T) = |s_1|^2 e^{\frac{j2\pi}{K}(m_1T + r_1T^2)} e^{\frac{j2\pi(2r_1lT)}{K}} + |s_2|^2 e^{\frac{j2\pi}{K}(m_2T + r_2T^2)} e^{\frac{j2\pi(2r_2lT)}{K}} + \cdots + \text{cross terms} \tag{6}$$

where the cross terms are of the form

$$s_p\bar{s}_q e^{\frac{j2\pi}{K}(m_pT + r_pT^2)} e^{\frac{j2\pi}{K}l(m_p - m_q + 2Tr_p) + \frac{j2\pi}{K}l^2(r_p - r_q)} \tag{7}$$