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Robust dimension reduction, fusion frames, and Grassmannian packings

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Abstract

We consider estimating a random vector from its measurements in a fusion frame, in presence of noise and subspace erasures. A fusion frame is a collection of subspaces, for which the sum of the projection operators onto the subspaces is bounded below and above by constant multiples of the identity operator. We first consider the linear minimum mean-squared error (LMMSE) estimation of the random vector of interest from its fusion frame measurements in the presence of additive white noise. Each fusion frame measurement is a vector whose elements are inner products of an orthogonal basis for a fusion frame subspace and the random vector of interest. We derive bounds on the mean-squared error (MSE) and show that the MSE will achieve its lower bound if the fusion frame is *tight*. We then analyze the robustness of the constructed LMMSE estimator to erasures of the fusion frame subspaces. We limit our erasure analysis to the class of tight fusion frames and assume that all erasures are equally important. Under these assumptions, we prove that tight fusion frames consisting of equi-dimensional subspaces have maximum robustness (in the MSE sense) with respect to erasures of one subspace among all tight fusion frames, and that the optimal subspace dimension depends on signal-to-noise ratio (SNR). We also prove that tight fusion frames consisting of equi-dimensional subspaces with equal pairwise chordal distances are most robust with respect to two and more subspace erasures, among the class of equi-dimensional tight fusion frames. We call such fusion frames *equi-distance tight fusion frames*. We prove that the squared chordal distance between the subspaces in such fusion frames meets the so-called *simplex bound*, and thereby establish connections between equi-distance tight fusion frames and *optimal Grassmannian packings*. Finally, we present several examples for the construction of equi-distance tight fusion frames.

Keywords: Distributed processing; Fusion frames; Grassmannian packings; LMMSE estimation; Packet encoding; Parallel processing; Robust dimension reduction; Subspace erasures

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1. Introduction

The notion of a *fusion frame* (or *frame of subspaces*) was introduced by Casazza and Kutyniok in [7] and further developed by Casazza et al. in [8]. A fusion frame for $\mathbb{R}^M$ is a finite collection of subspaces $\{\mathcal{W}_i\}_{i=1}^N$ in $\mathbb{R}^M$ such that there exist constants $0 < A \leqslant B < \infty$ satisfying

$$A\|\mathbf{x}\|^2 \leqslant \sum_{i=1}^{N} \|\mathbf{P}_i\mathbf{x}\|^2 \leqslant B\|\mathbf{x}\|^2, \quad \text{for any } \mathbf{x} \in \mathbb{R}^M,$$

where $\mathbf{P}_i$ is the orthogonal projection onto $\mathcal{W}_i$. Alternatively, $\{\mathcal{W}_i\}_{i=1}^N$ is a fusion frame if and only if

$$A\mathbf{I} \leqslant \sum_{i=1}^{N} \mathbf{P}_i \leqslant B\mathbf{I}. \tag{1}$$

The constants A and B are called (*fusion*) *frame bounds*. An important class of fusion frames is the class of *tight fusion frames,* for which A and B can be chosen to be equal and hence $\sum_{i=1}^N \mathbf{P}_i = A\mathbf{I}$. We note that the definition given in [7] and [8] for fusion frames applies to closed and weighted subspaces in any Hilbert space. However, since the scope of this paper is limited to non-weighted subspaces in $\mathbb{R}^M$, the definition of a fusion frame is only presented for this case.

A fusion frame can be viewed as a frame-like collection of low-dimensional subspaces. In frame theory, an input signal is represented by a collection of *scalars*, which measure the magnitudes of the projections of the signal onto frame vectors, whereas in fusion frame theory an input signal is represented by a collection of *vectors*, whose elements are the inner products of the signal and the orthogonal bases for the fusion frame subspaces. Similar to frames, fusion frames can be used to provide a redundant and non-unique representation of a signal. In fact, in many applications, where data has to be processed in a distributed manner by combining several locally processed data vectors, fusion frames can provide a more natural mathematical framework than frames. A few examples of such applications are as follows.

Distributed sensing. In distributed sensing, typically a large number of inexpensive sensors are deployed in an area to measure a physical quantity such as temperature, sound, vibration, pressure, etc., or to keep an area under surveillance for target detection and tracking. Due to practical and economical factors, such as low communication bandwidth, limited signal processing power, limited battery life, or the topography of the surveillance area, the sensors are typically deployed in clusters, where each cluster includes a unit with higher computational and transmission power for local data processing. Thus, a typical large sensor network can be viewed as a redundant collection of subnetworks forming a set of subspaces. The gathered subspace information is submitted to a central processing station for joint processing. Some references that consider fusion frames for distributed sensing are [16,17] and [9].

Parallel processing. If a frame system is simply too large to handle effectively (from the numerical standpoint), we can divide it into multiple small subsystems for more simple, and perhaps parallelizable, processing. By introducing redundancy, when splitting the large system, we can introduce robustness against errors due to failure of a subsystem. Fusion frames provide a natural framework for splitting a large frame system into smaller subsystems and then recombining the subsystems. The use of fusion frames for parallel computing has been considered in [1].

Packet encoding. In digital media transmission, information bearing source symbols are typically encoded into a number of packets and then transmitted over a communication network, e.g., the internet. The transmitted packet may be corrupted during the transmission or completely lost due to, for example, buffer overflows. By introducing redundancy in encoding the symbols, according to an error-correcting scheme, we can increase the reliability of the communication scheme. Fusion frames, as redundant collections of subspaces, can be used to produce a redundant representation of a source symbol. In the simplest form, we can think of each fusion frame measurement as a packet that carries some new information about the symbol. At the destination the packets can be decoded jointly to recover the transmitted symbol. The use of fusion frames for packet encoding is considered in [2].

The optimal reconstruction (in ℓ_2 norm sense) of a deterministic signal $\mathbf{x} \in \mathbb{R}^M$ from its fusion frame measurements is considered in [8]. In this paper, we consider the linear minimum mean-squared error (LMMSE) estimation (cf. [18, Chapter 8]) of a *random vector* $\mathbf{x} \in \mathbb{R}^M$ from its fusion frame measurements, in presence of additive white noise and subspace erasures. Each fusion frame measurement is a low-dimensional (smaller than M) vector whose elements