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Adaptive wavelet methods using semiorthogonal spline wavelets: Sparse evaluation of nonlinear functions

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Abstract

Enormous progress has been made in the construction and analysis of adaptive wavelet methods in the recent years. Cohen, Dahmen, and DeVore showed that such methods converge for a wide class of operator equations, both linear and nonlinear. Moreover, they showed that the rate of convergence is asymptotically optimal and that the methods are asymptotically optimally efficient. So far, these methods are based upon biorthogonal wavelets with compactly supported primal and dual functions. Semiorthogonal spline wavelets offer some quantitative advantages, namely small supports and good conditioning of the bases. On the other hand, the corresponding dual functions are globally supported so that they are ruled out for existing wavelet methods for nonlinear variational problems. In this paper, we focus on a core ingredient of adaptive wavelet methods for nonlinear problems, namely the adaptive evaluation of nonlinear functions. We present an efficient adaptive method for approximately evaluating nonlinear functions of wavelet expansions using semiorthogonal spline wavelets. This is achieved by modifying and extending a method for compactly supported biorthogonal wavelets by Dahmen, Schneider, and Xu. In order to do so, we introduce a new adaptive quasi-interpolation scheme, a corresponding prediction and a new decomposition. We give a complete analysis including an investigation of the complexity.

Keywords: Semiorthogonal wavelets; Nonlinear operators

1. Introduction

Enormous progress has been made in the construction and analysis of adaptive wavelet methods for the numerical solution of a wide class of linear and nonlinear variational operator equations. These operators include partial differential operators (also systems such as in the Stokes equation) and integral operators. It has been shown in [15,21,23] that appropriate adaptive wavelet methods converge and in [15,16,22] that the rate of convergence is optimal as compared with the best N-term approximation. So far, these adaptive Wavelet–Galerkin methods require the availability of biorthogonal wavelets where both primal and dual functions are compactly supported. This rules out semiorthogonal spline wavelets since the corresponding dual functions are globally supported.

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On the other hand, semiorthogonal spline wavelets have successfully been used in numerical simulations. These functions have some properties that make them particularly attractive for numerical purposes. First of all, scaling functions and wavelets are splines which allow for a fast evaluation. Second, the size of the support of semiorthogonal spline wavelets is smaller than the support of biorthogonal spline wavelets, so that the corresponding schemes are more local and also potentially (quantitatively) more efficient. Finally, the level-wise orthogonality results in a better quantitative stability. This means that the *condition number* of the wavelet basis $\Psi = \{\psi_\lambda\colon \lambda \in \nabla\}$ is smaller for semiorthogonal than for biorthogonal wavelets. Let us recall the definition of the condition number of a basis. A family $\Psi = \{\psi_\lambda\colon \lambda \in \nabla\}$ is a called *Riesz basis* for a Hilbert space X if it is complete in X and if the inequalities

$$c_\Psi \|\boldsymbol{d}\|_{\ell_2(\nabla)} \leqslant \Big\| \sum_{\lambda\in\nabla} d_\lambda \psi_\lambda \Big\|_X \leqslant C_\Psi \|\boldsymbol{d}\|_{\ell_2(\nabla)}, \quad \boldsymbol{d} = (d_\lambda)_{\lambda\in\nabla}, \tag{1.1}$$

are satisfied, with constants $0 < c_\Psi \leqslant C_\Psi < \infty$. The constants c_Ψ and C_Ψ are called *lower* and *upper Riesz bound*, respectively. The fraction

$$\rho_\Psi := \frac{C_\Psi}{c_\Psi} \tag{1.2}$$

of the smallest upper and the largest lower Riesz bound is called the *condition* of the Riesz basis. Obviously, this number gives a quantitative description of the stability of Ψ with respect to X. The optimal case obviously occurs when Ψ is an orthonormal basis, i.e. $\rho_\Psi = 1$. It is meanwhile well known that the larger ρ_Ψ is, the slower is the convergence of the adaptive wavelet method.

For numerical purposes we require in addition that the basis functions ψ_λ are compactly supported. With respect to these requirements Daubechies' orthonormal compactly support wavelets [28] on $L_2(\mathbb{R})$ would be the optimal choice. However, the use of these functions has some obstructions. Firstly, orthonormality might be a too severe restriction, e.g. when $\Omega \subseteq \mathbb{R}^n$ is complicated or when subspaces of $H_0^1(\Omega)$ have to be considered such as $H(\mathrm{div};\Omega)$ or $H(\mathrm{curl};\Omega)$ (where no compactly supported orthonormal wavelet basis exists). Moreover, these functions are not given by a closed formula but only in terms of a refinement equation. This makes issues like quadrature or point evaluation (even if possible exactly) possibly costly in an adaptive framework.

Hence, for both issues, piecewise polynomial functions would be convenient. For biorthogonal B-spline wavelets from [1,25], one can observe condition numbers (numerically) which are not satisfactory (see also Table 3 below). Several attempts have been made in order to optimize or precondition these bases, e.g. [3,26,29,31,35], which was quite successful in 1D but not optimal in higher dimensions.

As an alternative, we consider semiorthogonal spline wavelets here. These wavelets are spline functions, which are levelwise orthogonal and stable on each level. Therefore, Ψ has the Riesz bounds $c_\Psi = \min_j c_{\Psi_j}$ and $C_\Psi = \max_j C_{\Psi_j}$, where Ψ_j consists of all wavelets of the same level (or scale) j. Since stability is much easier to control on a single level, the commonly used examples of semiorthogonal wavelets posses good quantitative stability properties. The main reason why they have not been used so far in adaptive Wavelet–Galerkin methods for nonlinear variational problems lies in the fact that the dual system $\tilde{\Psi}$ is global, i.e., the dual wavelets $\tilde{\psi}_\lambda$ are globally supported. This means that the decomposition scheme relies on an infinite mask. It comes out somewhat surprising that this does not pose an obstacle as we show in this paper. It should be noted that semiorthogonal spline wavelets have already successfully been used in Galerkin methods for linear problems, e.g. [32,34].

For nonlinear variational problems, a core ingredient of an adaptive Wavelet–Galerkin method is the evaluation of nonlinear functions of a wavelet expansion. An efficient approximation method for this task is introduced by Dahmen, Schneider and Xu in [27] (from now on called *DSX-method*) for the case of biorthogonal *compactly* supported wavelets (see also [2,17] for subsequent investigations and corresponding tests). In this paper, we generalize the DSX-method to semiorthogonal spline wavelets. Roughly speaking, the DSX-method consists of 4 steps called *prediction*, *reconstruction*, *quasi-interpolation* and *decomposition*. It turns out that we have to introduce a new quasi-interpolation scheme for semiorthogonal spline wavelets. The analysis of this quasi-interpolation scheme will also lead to a partly new prediction.

This paper is organized as follows. In Section 2 we review preliminaries on semiorthogonal wavelets, adaptive Wavelet–Galerkin methods and recall the main ingredients of the DSX-method. The remainder of this paper is devoted to the modification of the DSX-method in [27] for semiorthogonal wavelets. In Section 3, we introduce uniform (nonadaptive) quasi-interpolation schemes, with emphasis on splines. These schemes are used in Section 4 to construct