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Confinement of the infrared divergence for the Mumford process ☆

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Abstract

The Mumford process X is a stochastic distribution modulo constant and cannot be defined as a stochastic distribution invariant in law by dilations. We present two expansions of X—using wavelet bases—in $X = X_0 + X_1$ which allow us to confine the divergence on the "small term" X_1 and which respect the invariance in law by dyadic dilations of the process.

Keywords: Mumford process; Infrared divergence; Wavelet bases; Self similarity

1. Introduction

The Mumford process has been introduced by Mumford and Gidas in [7] as the simplest process which can generate images. In this paper, Mumford and Gidas define the axioms that a stochastic process shall verify to generate images. Let us cite, for example,

(1) the scaling invariance which express the fact that an object seems bigger but does not change of form when one approaches it,
(2) the infinite divisibility which means that an image can be seen as the superposition of (less complex) independent images.

The Mumford process satisfies these axioms since it is a Gaussian stochastic process with stationary increments and invariant by dilations. Nevertheless, since it is Gaussian, it can only simulate clouds and not complex images.

This process is defined as a stochastic distribution modulo constants almost everywhere. It is known that it can be defined as a stochastic distribution, but with this definition, the property of scaling invariance is lost. Our point of view is to conserve this scaling invariance. We will see (in Section 2) that it is then impossible to define the Mumford process as a stochastic distribution, invariant in law by dilations. In particular, any expansion on a wavelet basis of the

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Sobolev space $\dot{H}^1(\mathbb{R}^2)$ leads to the phenomenon of infrared divergence and does not converge in the distributional sense.

Similarly to what has been done in [12] for the confinement of the infrared divergence of the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$, with $s - \frac{n}{2} \in \mathbb{N}$, our goal is to divide the Mumford process X in $X = X_0 + X_1$, where X_0 can be defined as a stochastic distribution and X_1 is "as small as possible." Moreover, we are looking for solutions which can be rapidly and robustly implemented (the robustness will be given by the unconditionality of the basis on which the processes are expanded).

We present in this paper two explicit solutions. The first one consists in writing, in the frequency domain, $\hat{X}(\xi, \omega)$ as the sum of a radial term and an anti-radial term. Expanding the terms on a suitable orthonormal basis, the infrared divergence is carried by the radial term (Section 3).

The second solution is based on the construction of an adapted basis, the wavelet basis with pseudo-constant (Section 4). It allows us to confine the infrared divergence on a smaller term than with the previous solution but the terms are now correlated (Section 5).

Let us just mention that there exists an orthonormal basis which provides us a confinement of the same order than the one given by the wavelet basis with pseudo-constant, but with decorrelated terms. But this 'ideal' solution is not constructive (the result can be found in [10]).

Remark. Matheron has given a systematic approach of processes with stationary increments defined on $\mathbb{R}^n$ in his paper [3]. He restricts its study to the case of random functions with continuous trajectories. They are then tested again atomic measures whose all moments of order less than or equal to k vanish (k fixed in $\mathbb{N}^n$). If Z is such a process and if λ is an atomic measure satisfying the vanishing moments condition, Matheron proves that there exists a random function Y, called the representation of Z, such that

$$Z(\lambda) = \int \lambda(\mathrm{d}x)\, Y(x, \omega), \tag{1}$$

where Y has continuous trajectories. Moreover, if X satisfies also (1) then $Y = X + \sum_{l \leqslant k} A^l x^l$, where A_l are random coefficients. It means that the representation is given modulo the random polynomials of degree less than or equal to k.

In the case of the Mumford process, the trajectories are not continuous and can not be tested again atomic measures but are tested again smooth functions with fast decay and with a vanishing integral. Our goal is then to find a representation $Y = Y_0 + Y_1$, where Y_1 can be tested again the functions of the Schwartz class.

Notations. We will denote by $\mathcal{S}_0(\mathbb{R}^2)$ the subspace of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ formed by the functions u satisfying

$$\int x^\alpha u(x)\, \mathrm{d}x = 0 \quad \forall \alpha \in \mathbb{N}^2,$$

and by $\mathcal{S}_0'(\mathbb{R}^2)$ its dual. This space is identified with $\mathcal{S}'(\mathbb{R}^2)/\mathcal{P}$.

Let us denote by $\dot{\mathcal{H}}^1(\mathbb{R}^2)$ the subspace of distributions f such that

$$R(f) := \left(\|\partial_{x_1} f\|_{L^2}^2 + \|\partial_{x_2} f\|_{L^2}^2\right)^{\frac{1}{2}} < \infty.$$

The homogeneous Sobolev space denoted by $\dot{H}^1(\mathbb{R}^2)$ is the quotient of $\dot{\mathcal{H}}(\mathbb{R}^2)$ with $\mathbb{C}$ and is equipped with the norm $\|\cdot\|_{\dot{H}^1} = R(\cdot)$. Its dual space, for the L^2-scalar product, is the homogeneous Sobolev space $\dot{H}^{-1}(\mathbb{R}^2)$.

2. Definition of the Mumford process

We will present the definitions of the complex and the real Mumford processes.

The complex Mumford process $X(x, \omega)$ is formally defined from the complex white noise $Z(x, \omega)$, for $x \in \mathbb{R}^2$, by

$$X(x, \omega) = \Lambda^{-1} Z(x, \omega), \tag{2}$$