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Letter to the Editor

Jump relations of the quadruple layer potential on a regular surface in three dimensions

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Abstract

The jump relations of the quadruple layer potential on a regular surface in three dimensions are derived. The jumps are shown to be proportional to the product of the density of the potential and the mean curvature of the underlying surface.

Keywords: Jump relation; Quadruple layer potential; Regular surface; Mean curvature

1. Introduction

With the recent advances in fast algorithms (see, for example, [2]) for solving integral equations resulting from potential theory, there is a renewed interest in the classical potential theory. As is well known, the jump relations of the single and double layer potentials play an important role in the classical potential theory. In [3], the jump relations of the quadruple layer potential on a curve in two dimensions are derived; and it is shown that the jumps are proportional to the product of the density of the potential and the curvature of the curve. In this note, we derive the jump relations of the quadruple layer potential on a regular surface in three dimensions and show that the jumps of the quadruple layer potential are proportional to the product of the density and the mean curvature of the underlying surface. The result is summarized in Theorem 3.8.

2. Analytical preliminaries

In this section, we collect some well-known facts from classic analysis to be used in the remainder of the paper.

2.1. Notation

We will denote by S a sufficiently smooth (say, at least twice continuously differentiable) regular surface in $\mathbb{R}^3$. When S is an open surface with boundary, we assume that S is an open set, i.e., S does not contain its boundary C.

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For a point $x \in S$, we denote the unit normal vector to S at x by $N(x)$. For a vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we will denote its length by $|x|$. Finally, for two vectors $x, y \in \mathbb{R}^3$, we denote their inner product by $\langle x, y\rangle$.

2.2. *Single, double, and quadruple layer potentials*

For any $x, t \in \mathbb{R}^3$ and $x \neq t$, we define the Green's function for the Laplace equation in $\mathbb{R}^3$ via the formula

$$G(x,t) = \frac{1}{|x-t|}. \tag{1}$$

Suppose now that S is a sufficiently smooth regular surface. For $t \in S$ we consider the directional derivatives of the function G with respect to t along the normal directions of S at t. It is easy to verify that

$$\frac{\partial G(x,t)}{\partial N(t)} = N(t)\cdot \nabla_t G(x,t) = \frac{\langle N(t), x-t\rangle}{|x-t|^3}, \tag{2}$$

$$\frac{\partial^2 G(x,t)}{\partial N(t)^2} = N(t)\cdot \nabla_t \nabla_t G(x,t)\cdot N(t) = \frac{3\langle N(t), x-t\rangle^2}{|x-t|^5} - \frac{1}{|x-t|^3}. \tag{3}$$

In the literature, (1)–(3) are often referred to as the single, double, and quadruple potentials, respectively. Suppose further that $\sigma : S \to \mathbb{R}$ is a sufficiently smooth function. We will refer to the functions given by the formulae

$$\int_S G(x,t)\cdot \sigma(t)\,\mathrm{d}t, \tag{4}$$

$$\int_S \frac{\partial G(x,t)}{\partial N(t)}\cdot \sigma(t)\,\mathrm{d}t, \tag{5}$$

$$\int_S \frac{\partial^2 G(x,t)}{\partial N(t)^2}\cdot \sigma(t)\,\mathrm{d}t, \tag{6}$$

as the single, double, and quadruple layer potentials, respectively.

2.3. *Finite part integrals on a regular surface in $\mathbb{R}^3$*

Finite part (also referred to as hypersingular) integrals on Euclidean spaces are well known and extensively used in mechanical engineering. Here we generalize the definition for the Euclidean spaces given by Samko (see Chapter 1, Section 5.3 in [4]) to define finite part integrals on a regular surface in $\mathbb{R}^3$.

Definition 2.1. Suppose that S is a sufficiently smooth regular surface. Suppose further that the function f is integrable in $S - D_\varepsilon(x)$ for all $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, where $D_\varepsilon(x) = \{t \in S \mid |t - x| < \varepsilon\}$. Then f is said to possess the *Hadamard property at x* if there exist constants a_k, b, and λ_k, possibly complex-valued, but $\mathrm{Re}(\lambda_k) > 0$, $k = 1, 2, \ldots, N$, such that

$$\int_{S-D_\varepsilon(x)} f(t)\,\mathrm{d}t = \sum_{k=1}^{N} a_k \cdot \varepsilon^{-\lambda_k} + b\cdot \log\frac{1}{\varepsilon} + I_0(\varepsilon), \tag{7}$$

where $\lim_{\varepsilon\to 0} I_0(\varepsilon)$ exists and is finite. In this case the *finite part* of the (divergent) integral $\int_{S-D_\varepsilon(x)} f(t)\,\mathrm{d}t$ is defined by the formula

$$\text{f.p.}\int_S f(t)\,\mathrm{d}t = \lim_{\varepsilon\to 0} I_0(\varepsilon). \tag{8}$$

Remark 2.1. When S is a flat surface embedded in $\mathbb{R}^2$, the above definition coincides with the conventional definition for finite part integrals in $\mathbb{R}^n$ given in [4].