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Riemannian geometry of the space of volume preserving immersions [☆]

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ABSTRACT

Given a compact manifold M and a Riemannian manifold N of bounded geometry, we consider the manifold $\mathrm{Imm}(M,N)$ of immersions from M to N and its subset $\mathrm{Imm}_\mu(M,N)$ of those immersions with the property that the volume-form of the pull-back metric equals μ. We first show that the non-minimal elements of $\mathrm{Imm}_\mu(M,N)$ form a splitting submanifold. On this submanifold we consider the Levi-Civita connection for various natural Sobolev metrics, we write down the geodesic equation for which we show local well-posedness in many cases. The question is a natural generalization of the corresponding well-posedness question for the group of volume-preserving diffeomorphisms, which is of importance in fluid mechanics.

1. Introduction

Let M be a compact connected (oriented) d-dimensional manifold, and let $(N,\bar{g})$ be a Riemannian manifold of bounded geometry. In this article we study Riemannian metrics on the space $\mathrm{Imm}_\mu(M,N)$ of all immersions from M to N that preserve a fixed volume form μ; i.e., those immersions f such that $\mathrm{vol}(f^*\bar{g}) = \mu$.

The interest in this space can be motivated from applications in the study of biological membranes, where the volume density of the surface remains constant during certain biological deformations. On the other hand there are similarities to mathematical hydrodynamics, as the space $\mathrm{Imm}_\mu(M,N)$ can be seen

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as a direct generalization of the group of all volume preserving diffeomorphisms. As a consequence the geodesic equations studied in Sect. 4 can be seen as an analogue of Euler's equation for the motion of an incompressible fluid. We will employ similar methods as Ebin and Marsden [10] to study the wellposedness of some of the equations that appear in the context of (higher order) metrics on $\mathrm{Imm}_\mu(M,N)$. Finally, the analysis of this article can be seen as a direct continuation of the analysis of Preston [22–24] for the motion and geometry of the space of whips and chains; these correspond to the choice $M = S^1$ or $M = [0,1]$ and $N = \mathbb{R}^2$. In Sect. 6 we will compare the results of this article with some of the results in these already better investigated situations.

We will consider the space $\mathrm{Imm}_\mu(M,N)$ as a subspace of the bigger space of all smooth immersions from M to N. Another interesting space that appears in this context is the space $\mathrm{Imm}_g(M,N)$ of all isometric immersions; i.e., all immersions that pull back $\bar{g}$ to a fixed metric g on M. Similarly, one can consider all these spaces in the context of embeddings as well. We have the following diagram of inclusions:

$$\begin{array}{ccccc} \mathrm{Imm}_g(M,N) & \hookrightarrow & \mathrm{Imm}_\mu(M,N) & \hookrightarrow & \mathrm{Imm}(M,N) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Emb}_g(M,N) & \hookrightarrow & \mathrm{Emb}_\mu(M,N) & \hookrightarrow & \mathrm{Emb}(M,N) \end{array}$$

Here $\mathrm{Emb}_g(M,N)$ and $\mathrm{Emb}_\mu(M,N)$ are defined similar as for the bigger spaces of immersions. We will concentrate in this article on the space $\mathrm{Imm}_\mu(M,N)$ (resp. $\mathrm{Emb}_\mu(M,N)$) and we plan to consider the geometry of the space of isometric immersions (embeddings) in future work.

In the article [19] it has been shown that the space $\mathrm{Emb}_\mu^\times(M,N)$ is a smooth tame splitting submanifold of the space of all smooth embeddings $\mathrm{Emb}(M,N)$, where the elements of the spaces $\mathrm{Emb}_\mu^\times(M,N)$ are assumed to have nowhere vanishing second fundamental form. The choice of this space is not very fortunate for our purposes for various reasons; e.g., in the case of closed surfaces in $\mathbb{R}^3$ this condition restricts to convex surfaces only. Thus, as a first step, we want to get rid of that additional condition and show a similar statement for the spaces in the above diagram. As in [19], the proof of this statements will be an application of the Nash–Moser inverse function theorem. However, we will have to consider a different splitting of the tangent space. The proof of these statements will be given in Sect. 3. We will still be forced to require the immersions to be not minimal; i.e., they do not have an everywhere vanishing mean curvature. In the case of embeddings into $\mathbb{R}^3$, the absence of compact minimal embeddings already shows that this is only a weak restriction. For the space of all volume preserving embeddings the submanifold result has been shown in [13], using a different method of proof.

In the second part of this article we will equip the space $\mathrm{Imm}(M,N)$ with the family of reparametrization invariant Sobolev metrics as introduced in [5,6]:

$$G_f(h,k) = \int_M \bar{g}((1+\Delta)^l h, k)\mu, \qquad l \in \mathbb{N}.$$

Here Δ denotes the Bochner–Laplacian of the pullback metric $g = f^*\bar{g}$. See also [3] for an overview on various metrics on spaces of immersions. In this article we will be interested in the induced metric of these metrics on the submanifold $\mathrm{Imm}_\mu(M,N)$. In particular we will discuss the orthogonal projection from $T\,\mathrm{Imm}(M,N)$ to $T\,\mathrm{Imm}_\mu(M,N)$ with respect to these metrics, we will consider the induced geodesic equation on the submanifold, and we will give sufficient conditions on the order l to ensure local well-posedness of the corresponding geodesic equations.

We will conclude the article with the two special cases of volume preserving diffeomorphisms ($M = N$) and constant speed parametrized curves ($M = S^1$, $N = \mathbb{R}^2$).