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Transformations and coupling relations for affine connections

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The statistical structure on a manifold $\mathfrak{M}$ is predicated upon a special kind of coupling between the Riemannian metric g and a torsion-free affine connection ∇ on $T\mathfrak{M}$, such that ∇g is totally symmetric, forming, by definition, a "Codazzi pair" $\{\nabla, g\}$. In this paper, we first investigate various transformations of affine connections, including additive translation by an arbitrary $(1, 2)$-tensor K, multiplicative perturbation through an arbitrary invertible operator L on $T\mathfrak{M}$, and conjugation through a non-degenerate bilinear form h. We then study the Codazzi coupling of ∇ with h and its coupling with L, and the link between these two couplings. We introduce, as special cases of K-translations, various transformations that generalize traditional projective and dual-projective transformations, and study their commutativity with L-perturbation and h-conjugation transformations. Our framework allows affine connections to carry torsion, and we investigate conditions under which torsions are preserved by the various transformations mentioned above. In addition to reproducing some known results regarding Codazzi transformations, conformal-projective transformations, etc., we extend many of these geometric relations, and hence obtain new geometric insights, for the general case of a non-degenerate bilinear form h (not required to be a symmetric form g) in relation to an affine connection with possibly non-vanishing torsion. In particular, we provide a generalization to the conformal-projective transformation of $\{\nabla, g\}$ which preserves their Codazzi coupling. Our systematic approach establishes a general setting for the study of information geometry based on transformations and coupling relations between affine connections.

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1. Introduction

On the tangent bundle $T\mathfrak{M}$ of a differentiable manifold $\mathfrak{M}$, one can introduce two separate structures: an affine connection ∇ and a Riemannian metric g. The coupling of these two structures has been of great interest to, say, affine geometers and information geometers. When coupled, $\{\nabla, g\}$ is called a Codazzi pair, e.g., [17,20], which is an important concept in affine hypersurface theory, e.g., [18,12], statistical manifolds

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[8], and related fields. To investigate the robustness of the Codazzi structure, one would perturb the metric and the affine connection, and examine whether, after perturbation, the resulting metric and connection will still maintain Codazzi coupling [15].

The Codazzi transformation is a useful concept that combines a projective transformation of an affine connection and a conformal transformation of the Riemannian metric so that the Codazzi coupling of the pair $\{\nabla, g\}$ is preserved. This transformation is specified by an arbitrary function that transforms both the metric and the connection, see [20]. The concept of conformal-projective transformation [7] generalized Codazzi transformation to the case using two arbitrary functions. A natural question to ask is whether there are more general transformations of the metric and of the connection that preserve the Codazzi coupling. In this paper, we answer this question in the affirmative by providing a generalization to the conformal-projective transformation. The second goal of this paper is to investigate the role of torsion in affine connections and their transformations. Research on this topic is isolated, and its general importance has not yet been appreciated.

Our paper starts by collecting various results on transformations on affine connection and classifying them through one of the three types, L-perturbation, h-conjugation, and the more general K-translation. They correspond to transforming ∇ via a $(1,1)$-tensor, $(0,2)$-tensor, or $(1,2)$-tensor, respectively. We then investigate the interactions between these transformations, based on known results but generalizing them to more arbitrary and less restrictive conditions. We will show how a general transformation of a non-degenerate bilinear form and a certain transformation of the connection are coupled; here transformation of a connection can be through L-perturbation, h-conjugation, or K-translation which specializes to various projective-like transformations. We will show how they are linked in the case when they are Codazzi coupled to the same connection ∇. The outcomes are depicted in commutative diagrams as well as stated as theorems.

The interaction between the projective structure and the conformal structure has been of great interest to information geometry. From the well-understood projective transformation and projective equivalence of affine connections, researchers have introduced, progressively, the notions of dual-projective transformation [5], α-conformal transformation [6] with $\alpha = -1$ describing projective transformation (and hence Codazzi transformation of the metric-connection pair) and $\alpha = 1$ describing dual-projective transformation, respectively, and conformal-projective transformation [7] which encompasses all previous cases. Recall that two statistical manifolds $(\mathfrak{M}, \nabla, g)$ and $(\mathfrak{M}, \nabla', g')$ are called [6] *α-conformally equivalent*, $\alpha \in \mathfrak{R}$, if there exists a function ϕ such that

$$g'(X,Y) = e^{\phi} g(X,Y),$$
$$\nabla'_Y X = \nabla_Y X - \frac{1+\alpha}{2} g(X,Y)\,\mathrm{grad}_g\,\phi + \frac{1-\alpha}{2}\{d\phi(X)Y + d\phi(Y)X\},$$

where $\mathrm{grad}_g\,\phi$ is the gradient vector field of ϕ with respect to g, namely, $g(X, \mathrm{grad}_g\,\phi) \equiv X(\phi) \equiv d\phi(X)$ for an arbitrary vector field X on $\mathfrak{M}$. When $\alpha = -1$, it describes projective equivalency. When $\alpha = 1$, it describes dual-projective equivalency. It is easily seen [22] that, for $\alpha, \beta \in \mathfrak{R}$,

(i) $(\mathfrak{M}, \nabla, g)$ and $(\mathfrak{M}, \nabla', g')$ are α-conformally equivalent iff $(\mathfrak{M}, \nabla^*, g)$ and $(\mathfrak{M}, \nabla'^*, g')$ are $(-\alpha)$-conformally equivalent.
(ii) If $(\mathfrak{M}, \nabla, g)$ and $(\mathfrak{M}, \nabla', g')$ are α-conformally equivalent, then $(\mathfrak{M}, \nabla^{(\beta)}, g)$ and $(\mathfrak{M}, \nabla'^{(\beta)}, g')$ are $(\alpha\beta)$-conformally equivalent.

Moreover, two statistical manifolds $(\mathfrak{M}, \nabla, g)$ and $(\mathfrak{M}, \nabla', g')$ are said to be *conformally-projectively equivalent* [7] if there exist two functions ϕ and ψ such that

$$g'(X,Y) = e^{\phi+\psi} g(X,Y),$$
$$\nabla'_Y X = \nabla_Y X - g(X,Y)\,\mathrm{grad}_g\,\psi + \{d\phi(X)Y + d\phi(Y)X\}.$$

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