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Generalized quasi Yamabe gradient solitons

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We prove that a nontrivial complete generalized quasi Yamabe gradient soliton (M^n, g) must be a quasi Yamabe gradient soliton on each connected component of M and that a nontrivial complete locally conformally flat generalized quasi Yamabe gradient soliton has a special warped product structure.

1. Introduction

A complete Riemannian manifold (M^n, g), $n \geq 3$, is a *generalized quasi-Einstein manifold*, if there exist three smooth functions f, μ and β on M such that

$$Ric + \nabla^2 f - \mu df \otimes df = \beta g,$$

where Ric and ∇^2 denotes, respectively, the Ricci tensor and Hessian of the metric g. This concept, introduced by Catino in [5], generalizes the m-quasi-Einstein manifolds (see, for instance [1,12]). Inspired by [5], we will introduce a class of Riemannian manifolds (see [6]).

A complete Riemannian manifold (M^n, g), $n \geq 3$, is a *generalized quasi Yamabe gradient soliton* (GQY manifold), if there exist a constant λ and two smooth functions, f and μ, on M, such that

$$(R - \lambda)g = \nabla^2 f - \mu df \otimes df \quad (1.1)$$

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where R denotes the scalar curvature of the metric g and df is the dual 1-form of ∇f. In a local coordinates system, we have

$$(R - \lambda)g_{ij} = \nabla_i \nabla_j f - \mu \nabla_i f \nabla_j f. \tag{1.2}$$

When f is a constant function, we say that (M^n, g) is a *trivial* generalized quasi Yamabe gradient soliton. Otherwise, it will be called *nontrivial*. In the special case where μ is constant, we will have a *quasi Yamabe gradient soliton*. Let us point out that if $\mu = 0$, (1.1) becomes the fundamental equation of *gradient Yamabe soliton*.

The Yamabe flow was introduced in [9] as an attempt to solve the Yamabe problem. Yamabe solitons are self-similar solutions of the Yamabe flow [4,6,8,9]. For $\lambda = 0$ the Yamabe soliton is *steady*, for $\lambda < 0$ is *expanding*, and for $\lambda > 0$ is *shrinking*. It has been known (see [14]) that a compact Yamabe soliton has constant scalar curvature, thus trivial. Daskalopoulos and Sesum [8] proved that locally conformally flat gradient Yamabe solitons with positive sectional curvature are rotationally symmetric. Then in [4], they proved that a gradient Yamabe soliton admits a warped product structure without any additional hypothesis. They also proved that a locally conformally flat gradient Yamabe solitons has a more special warped product structure. Inspired by the Generalized quasi-Einstein metrics (see [5,12]), they started to consider the quasi Yamabe gradient solitons (see [10,11,15]). In [10], they introduced the concept of quasi Yamabe gradient soliton and showed that locally conformally flat quasi Yamabe gradient solitons with positive sectional curvature are rotationally symmetric. Moreover, they proved that a compact quasi Yamabe gradient soliton has constant scalar curvature. Leandro [11] investigated the quasi Yamabe gradient solitons on four-dimensional case and proved that half locally conformally flat quasi Yamabe gradient solitons with positive sectional curvature are rotationally symmetric. And he proved that half locally conformally flat gradient Yamabe solitons admit the same warped product structure proved in [4]. Wang [15] gave several estimates for the scalar curvature and the potential function of the quasi Yamabe gradient solitons. He also proved that a quasi Yamabe gradient solitons carries a warped product structure. In [6], they define and study the geometry of gradient Einstein-type manifolds, in particular, they proved that the GQY manifolds has a warped product structure. This metric generalizes the GQY manifolds.

In this paper, we first prove the following result.

Theorem 1.1. *Let (M^n, g), $n \geq 3$, be a nontrivial complete generalized quasi Yamabe gradient soliton satisfying (1.1). Then, μ must be constant on each connected component of M.*

Therefore we can announce the following result analogous to the Proposition 2.1 in [4] (we also recommend [8,15]).

Theorem 1.2. *Let (M^n, g) be a nontrivial complete connected generalized quasi Yamabe gradient Yamabe soliton, satisfying the GQY equation (1.1), and let $\Sigma_c = f^{-1}(c)$ be a regular level surface. Then*

(1) *The scalar curvature R and $|\nabla f|^2$ are constants on Σ_c.*
(2) *The second fundamental form of Σ_c is given by*

$$h_{ab} = \frac{H}{n-1} g_{ab}. \tag{1.3}$$

(3) *The mean curvature $H = (n-1)\frac{(R-\lambda)}{|\nabla f|}$ is constant on Σ_c.*
(4) *In any open neighborhood $U_\alpha^\beta = f^{-1}\big((\alpha, \beta\big)$ of Σ_c in which f has no critical points, the GQY metric g can be expressed as*